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ABSTRACT

This student manual consists of exercises and investigations designed to supplement existing units and materials written for the Thirteen College Curriculum Program (TCCP) course. Many sections can be used for independent work. This 174-page manual is divided into seven parts: (1) recognizing and representing number patterns; (2) sets and logic; (3) functions, relations and graphing; (4) tools and concepts; (5) computer BASIC; (6) probability; and (7) trigonometry. (JW)

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— PROBLEM BOOK —

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DEVELOPED BY THE INSTITUTE FOR SERVICES TO EDUCATION
in conjunction with
THIRTEEN COLLEGE CONSORTIUM
THE FIVE COLLEGE CONSORTIUM
EIGHT COLLEGE CONSORTIUM
AND THE CONSORTIUM FOR CURRICULUM CHANGE

2

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The Institute for Services to Education was incorporated as a non-profit organization in 1965 and received a basic grant from the Carnegie Corporation of New York. The organization is founded on the principle that education today requires a fresh examination of what is worth teaching and how to teach it. ISE undertakes a variety of educational tasks, working cooperatively with other educational institutions, under grants from government agencies and private foundations. ISE is a catalyst for change. It does not just produce educational materials or techniques that are innovative; it develops, in cooperation with teachers and administrators, procedures for effective installation of successful materials and techniques in the colleges.

ISE is headed by Dr. Elias Blake, Jr., a former teacher and is staffed by college teachers with experience in working with disadvantaged youth and Black youth in educational settings both in predominantly Black and predominantly white colleges and schools.

ISE's Board of Directors consists of persons in the higher education system with histories of involvement in curriculum change. The Board members are:

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- ABOUT THE THIRTEEN COLLEGE CURRICULUM PROGRAM -

From 1967 to the present, ISE has been working cooperatively with the Thirteen College Consortium in developing the Thirteen College Curriculum Program. The Thirteen College Curriculum Program is an educational experiment that included developing new curricular materials for the entire freshman year of college in the areas of English, mathematics, social science, physical science, and biology and two sophomore year courses, humanities and philosophy. The program is designed to reduce the attrition rate of entering freshman through well thought-out, new curricular materials, new teaching styles, and new faculty arrangements for instruction. In addition, the program seeks to alter the educational patter of the institutions involved by changing blocks of courses rather than by developing single courses. In this sense, the Thirteen College Curriculum Program is viewed not only as a curriculum program with a consistent set of academic goals for the separate courses, but also as a vehicle to produce new and pertinent educational changes within the consortium institutions. At ISE, the program is directed by Dr. Frederick S. Humphries, Vice-President. The curricular developments for the specific courses and evaluation of the program are provided by the following persons:

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In addition, Miss Patricia Parrish serves as general editor of the curriculum materials as well as the Assistant to the Director. Mrs. Joan Cooke is Secretary to the Executive Director.

The curriculum staff is assisted in the generation of new educational ideas and teaching strategies by teachers in the participating colleges and outside consultants. Each of the curriculum areas has its own advisory committee, with members drawn from distinguished scholars in the field but outside the program.

The number of colleges participating in the program has grown from the original thirteen of 1967 to nineteen in 1970. The original thirteen colleges are:

Alabama A and M University	Huntsville, Alabama
Bennett College	Greensboro, North Caroline
Clark College	Dallas, Texas
Florida A and M Universty	Atlanta, Georgia
Jackson State College	Tallahassee, Florida
Lincoln University	Lincoln University, Pennsylvania
Norfolk State College	Norfolk, Virginia
North Carolina A and T State University	Greensboro, North Caroline
Southern University	Baton Rouge, Louisiana
Talladega College	Talladega, Alabama
Tennessee State University	Nashville, Tennessee
Voorhees College	Denmark, South Carolina

A fourteenth college joined this consortium in 1968, although it is still called the Thirteen-College Consortium. The fourteenth member is:

Mary Holmes Junior College	West Point, Mississippi

In 1970, five more colleges joined the effort although linking up as a separate consortium. The members of the Five-College Consortium are:

Elizabeth City State University	Elizabeth City, North Carolina
Langston University	Langston, Oklahome
Southern University at Shreveport	Shreveport, Louisiana
Saint Augustine's College	Raleigh, North Carolina
Texas Southern University	Houston, Texas

In 1971, eight more colleges joined the curriculum development effort as another consortium. The member schools of the Eight College Consortium are:

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Alcorn A and M College	Lorman, Mississippi
Bethune-Cookman College	Daytona Beach, Florida
Grambling College	Grambling, Louisiana
Jarvis Christian College	Hawkins, Texas
LeMoyne-Owen College	Memphis, Tennessee
Southern University in New Orleans	New Orleans, Louisiana
University of Maryland, Eastern Shore	Princess Anne, Maryland
Virginia Union University	Richmond, Virginia

Seven additional colleges created still another consortium in 1972, entitled the Consortium for Curricular Change. These colleges are:

Coppin State College	Baltimore, Maryland
Huston-Tillotson College	Austin, Texas
Lincoln University	Jefferson City, Missouri
Mississippi Valley State College	Itta Bena, Mississippi
Shaw College	Detroit, Michigan
Bowie State College	Bowie, Maryland
Livingstone College	Salisbury, North Carolina

The Thirteen-College Curriculum Program has been supported by grants from:

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The Carnegie Corporation
The Esso Foundation

PREFACE

This student manual was written and compiled by teachers of the Thirteen College Curriculum Program (TCCP) with technical assistance provided by the mathematics staff of the Institute for Services to Education.

The manual is a book of exercises and investigations designed to supplement already existing mathematics units and other materials written for the TCCP course. However, many of the sections are sufficiently self-contained so that a student can use them for independent work.

The manual is divided into seven parts; each part is separated into sections (lessons) which are distinguished by Roman numerals. Most of the sections should be preceded by classroom demonstrations or discussions since the major part of this manual presents problems that are related to topics treated in written TCCP demonstrations.

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- RECOGNIZING AND REPRESENTING NUMBER PATTERNS -

I. NUMBER CONFIGURATIONS

a	b	c	d
e	f	g	h

(1)

	c	
a	b	d
e	f	g
	h	

(2)

Replace each letter in diagram 1 (then 2) above with a counting number 1 through 8 in such a way that no two letters are replaced by the same number and that no two consecutive numbers will be in connecting (adjacent) squares.

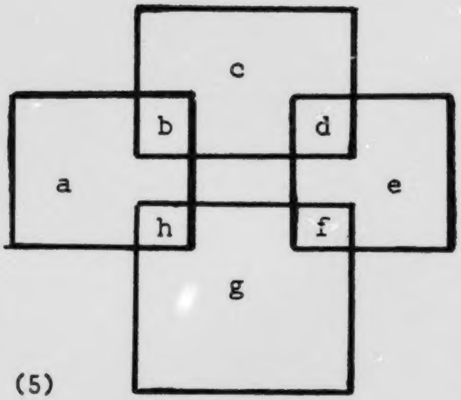
a	b	c
d	e	f
g	h	i

(3)

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

(4)

Replace each letter in diagram 3 (then 4) above with a counting number 1 through 9 (then 16) in such a way that no two letters are replaced by the same number and that the sum of the numbers in each row is the same as the sum of the numbers in any other row, column or diagonal.



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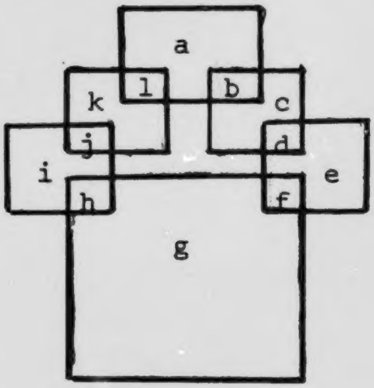


(6)

Replace each letter in diagram 5 (then 6) above, with a counting number 1 through 8 (then 12) in such a way that no two letters are replaced by the same number and that the sum of the numbers in each square is the same as the sum of the numbers in any of the other squares.

Develop a strategy for solving the following puzzles.

1. Soma Cube
2. Triangle Peg Puzzle
3. Instant Insanity
4. Block Puzzle

As you experiment with the puzzle:

1. List rules that will help you win.
2. Test your rules to see if they work.

After you have solved the puzzle, determine a winning strategy.

II. COUNTING GAMES

A. One to Twenty Counting Game

This game involves two players who use the first 20 counting numbers.

1. The first player selects the first counting number (1) or the first and second counting numbers (1, 2).
2. The second player selects the successor to the last number selected by the first player and, if he desires, the successor to his own number.
3. This continues until one of the players selects the number twenty; this player is the winner.
4. What is a winning strategy for this game? Can the first player always win, if he plays correctly?
5. Play the game with the first 19 counting numbers; the first 18 counting numbers. In which of these games can the first player always win if he plays correctly?
6. Design a similar game using powers of 2.

B. Another Counting Game

This game involves two players who use only the first 10 counting numbers.

1. The first player selects any subset of consecutive numbers that contains 1 but does not contain 10.
2. The second player selects any subset of consecutive numbers which contains the successor of the last number selected by the first player but such that the number of numbers selected does not exceed twice the number selected by the first player.
3. This continues until a player selects the number 10; this player is the winner.
4. What is a winning strategy for this game?

III. PRIME NUMBERS AND DIVISORS

1. What are the divisors of 6? 10? 25? 8? 24?

2. How would you define divisor of a positive integer?

3. How many positive integers between 1 and 50 have exactly one divisor?

4. How many positive integers between 1 and 50 have exactly two divisors? List some of these. Is there a name for these numbers? (Primes)

5. Are there other numbers between 1 and 50 which are primes? How about 1, itself?

6. How would you define prime number? What are the non-prime numbers called?

7. Write down all the positive integers from 1 to 50. To begin with, strike out the number 1. Next, strike out all of the multiples of 2 in your list except the number 2 itself. What type of numbers are you left with? Strike out the multiples of 3 in your list, other than 3 itself. What is the next prime after 3? Strike out the multiples of that prime number, except the prime itself. How far must this process be continued before the numbers remaining in your list consist of all the prime numbers from 1 to 50?

8. If your original list had included the positive integers from 1 to 100, how far would the process in Question 7 have to be continued so that you would have all the primes from 1 to 100? How about 1 to 1000? 1 to n?

9. Are there any numbers between 1 and 50 with exactly three divisors? List them. Now list the first four prime numbers. Any connection? Can you write a general form for positive integers having exactly three divisors? Does this hold for numbers greater than 50?

10. What are the numbers between 1 and 50 with exactly four divisors? Write these numbers in terms of their prime divisors, e.g., $6 = 2 \times 3$, $8 = 2^3$. Can you guess what the general forms of such numbers are? Does this hold for numbers greater than 50?

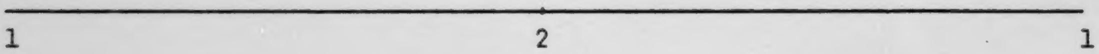
11. Can you find a number between 1 and 50 with exactly five divisors? Can you find another? Can you find one larger than 50?

12. In answering the questions in the exercises above, you have been actually laying the ground work for the completion of the following table. See how far you can go in filling in the rest of the table. Ultimately, try to fill in the nth row.

No. of divisors	Examples	General form
2	2,3,5,7	p (p is a prime)
3	4, 9, 25	p^2
4	6, 8	p^3, pq
5	16, 81	p^4
6		
7		
.		
.		
.		
n		

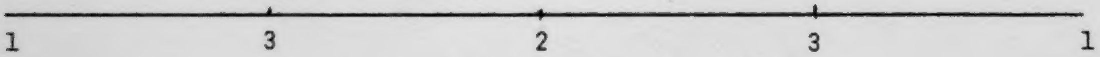
IV. SUBDIVIDING A LINE (EULER'S ϕ-FUNCTION)

Instructions: Take a sheet of paper and draw a straight line the length of the paper. Place the numeral 1 at each end of the line. Now, bisect the line and place the numeral 2 at the midpoint.



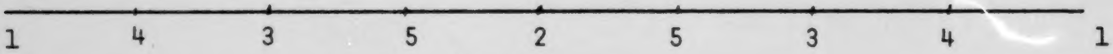
First Subdivision

Next, bisect each of the resulting two segments and place at the midpoint of each new segment the sum of the numbers at the end points of the segment.



Second Subdivision

Continue to subdivide the line in this manner, placing at the midpoint of each new segment the sum of the numbers at the endpoints of the segment.



Third Subdivision

1. After you have subdivided the line seven times, how many times does the number 2 appear on the line? How many times does 3 appear? Will any other 2's or 3's appear in additional subdivisions?
2. Continue to subdivide the line labeling the midpoints as above. Record the number of 2's, 3's, 4's, etc. that appear on the line.
3. Can you find some relationship between a given number and the frequency of its occurrences on the line? First try to establish the relationship for primes: 2, 3, 5, 7, 11, . . .

4. Try to complete the following table making as many subdivisions as necessary.

Number	Frequency
2	1
3	2
4	
5	
.	
.	
.	
n	

Hint: First consider special cases such as primes, powers of primes, etc., and search for a pattern.

V. LINE FRACTIONS

The fractions chart below was constructed by using a ruler, a sharpened pencil, and a piece of finely delineated graph paper (K & E Semi-Logarithmic 46 4530, 1 cycle x 60 divisions). On the 2nd line, the midpoint was found; the 3rd line was divided into thirds; the 4th line into fourths; etc. For some of the lines, such as the 7th, we approximated the subdivision, and then used the ruler to mark off this approximation across the rest of the line.

Make your own fractions chart (or complete the one below). Try to subdivide 15 or 20 lines. You might want to omit difficult subdivisions such as those involving 7ths, 11ths, 13ths, 17ths, etc.

Try to answer the following questions by referring to the chart.

1. Which is larger: 1/3 or 3/8? 3/5 or 4/7? 5/8 or 2/3? 3/7 or 4/9? 2/5 or 4/10? 1/2 or 5/10?

2. What is the numerator of the fraction ?/16 which is between 1/3 and 1/6? Is there a fraction with denominator 18 between 3/5 and 4/7? Find a fraction between 3/5 and 4/7.

3. What are some fractions which are equivalent to 1/3? to 1/6? to 2/3?

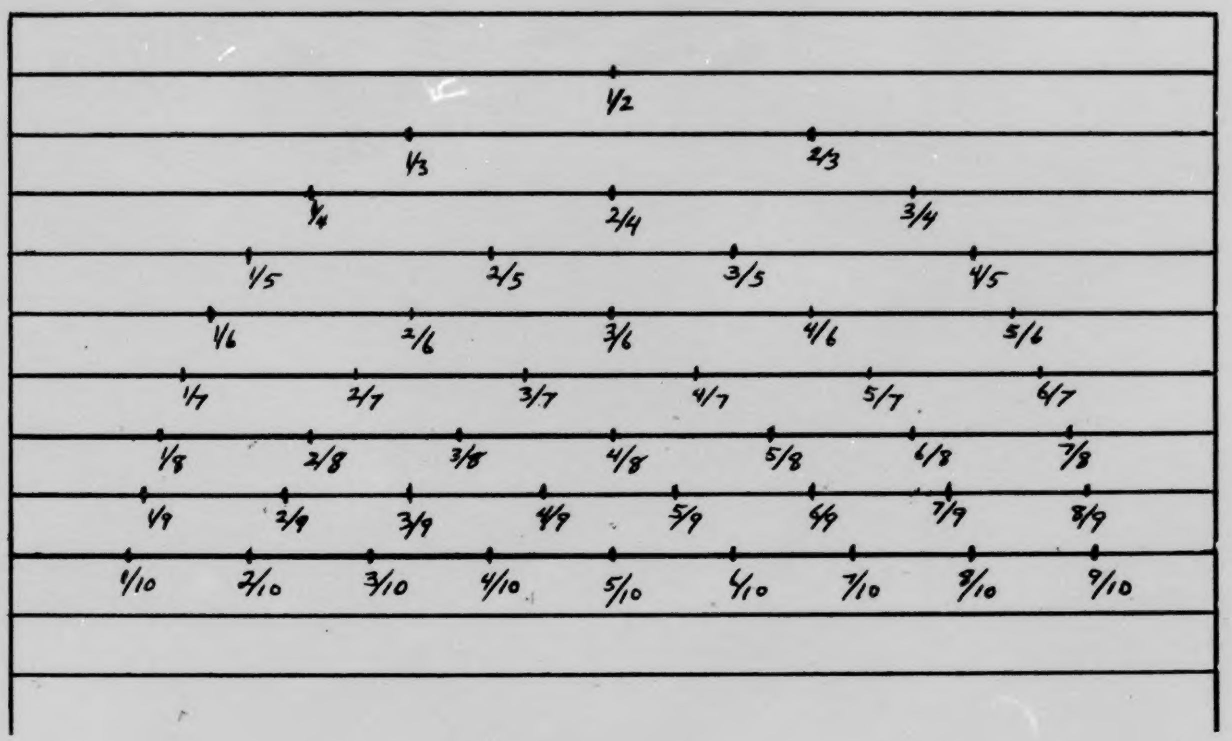
4. Make a long list of fractions equivalent to 1/3; what criteria must a fraction satisfy if it is to be in this list? Can you find a fraction with denominator 35 in your list? With denominator 32?

5. Repeat exercise 4 for other fractions, such as 1/4, 1/5, 2/3, etc.

6. Can you use the chart to figure out a way of adding the fractions 1/4 and 1/3? 1/3 and 2/5? 3/4 and 1/6? 3/5 and 3/4? Write a general rule.

7. Can you pick out any "diagonals" on your chart? What observations can you make about the fractions along a diagonal? Are any of these diagonals straight lines? Why? What kinds of curves are they?

FRACTIONS CHART


1/2
1/3 2/3
1/4 2/4 3/4
1/5 2/5 3/5 4/5
1/6 2/6 3/6 4/6 5/6
1/7 2/7 3/7 4/7 5/7 6/7
1/8 2/8 3/8 4/8 5/8 6/8 7/8
1/9 2/9 3/9 4/9 5/9 6/9 7/9 8/9
1/10 2/10 3/10 4/10 5/10 6/10 7/10 8/10 9/10

VI. REAL NUMBERS INTRODUCTION

A. Fractions and Whole Numbers

1. Are there any whole numbers that can be written as fractions?

2. Are there any fractions which can be written as whole numbers?

3. Are there any whole numbers which cannot be written as fractions? Are there any fractions which cannot be written as whole numbers?

4. When can a fraction [whole number] be written as a whole number [fraction]?

5. When is a fraction greater than 1? 2? 3? ... n?

6. When is a fraction less than 1? 2? 3? ... n?

7. When is a fraction between 1 and 2? 2 and 3? 3 and 4? ... n and n + 1?

B. Fractions and Decimals

1. Can you think of any fractions which can be written as decimals? How do you write a fraction in decimal form?

2. Can you find any decimals that can be written as fractions?

3. Write decimals for the fractions 1/2, 1/3, 1/4, 1/5, ... 1/20.

4. Notice that some of the fractions in 3 have decimal expansions which "repeat," such as:

   1/11 = .090909 ...,

   and some have expansions which do not repeat or which "terminate," such as:

   1/4 = .25.

Give several examples of fractions that have repeating decimals expansions. Is there an easy method or procedure for determining which fractions have repeating decimals? Look at several examples.

Hint: Consider the prime factors of the denominators of the fractions.

5. The fraction 1/7 has a repeating decimal; what about 2/7? 3/7? 4/7? etc. The fraction 1/8 terminates; what about 2/8, 5/8? Note that 1/12 repeats; what about 3/12? 5/12? 9/12?

6. What is the fraction for .555 ...? .666 ...? .999 ...? Hint: Recall that 1/9 = .111 ....

7. What fraction equals .2525 ...? .3737 ...? .362362 ...? Can you write these decimals as fractions whose denominator involve only the digit 9? e.g., .3737 ... = 362/999. Write a general rule.

C. Non-repeating Decimals

1. Are there any decimals which are non-repeating? How about .31311311131111 ...? .101101110111 ...?

2. Find a number whose square equals 2. Can you approximate such a number using decimals? Is the number larger than 1.5? 1.45? Is it less than 1.40? 1.42? Does this number have a repeating decimal expansion? Can it be written as a fraction?

3. If two numbers have repeating decimals, does their sum have a repeating decimal? How about the difference, product and quotient of the two numbers? Take specific examples.

4. If two numbers have non-repeating decimals, does their sum (product, difference, quotient) necessarily have a non-repeating decimal? Test your answer using:

   .39339333933339 ...
   .41441444144441 ...

   and other examples.

5. If one number has a repeating decimal and a second number has a non-repeating decimal what can you say about the sum (difference, product, quotient) of the two numbers, repeating decimal or non-repeating? Try several examples.

VII. BASE NUMERATION

A. Counting Money

1. Consider the following denominations of money: 1¢, 10¢, $1, $10, $100.

How much money would you have if you were given:

3 - $100 bills
5 - $10 bills
3 - $1 bills
4 - dimes
2 - pennies?

2. Suppose the amounts of $100, $10, $1, dimes and pennies that you have are 7, 2, 9, 6 and 1, respectively. How much do you have?

3. Do you notice a pattern in 1 and 2? Will this always work? What if the amounts of $100, $10, $1, dimes and pennies are 3, 4, 12, 11, 3? 8, 13, 5, 6, 11? How much do you have? How do these examples differ from the first two? Why?

4. Construct several such examples and make general observations.

B. Decimal System

Write down the numerals from 0 to 25.

1. Why do you use one digit to represent some of the numbers and two digits to represent others? When will you use three digits? Four digits? etc.

2. What kind of numeration system is this (Decimal)? Why is it called a "decimal system"?

3. What does 10 mean? 100? 1000? 1111? 132? Can you represent these numbers using only operations on 10?

C. Binary System

If we had a system with only two different digits, what should we call such a system? (Binary) What should the two symbols be?

Write the binary numbers that correspond to the numbers 0 to 25 in the decimal system.

Decimal	0	1	2	3	4	5	6	7	8 9 10 11	. . .
Binary	0	1	10	11	100			111	. . . .	. . .

D. Other Bases

The decimal system, having only 10 different digits, is sometimes referred to as "base 10"; the binary system as base 2.

1. Write the base 10 numbers 0 to 25 in base 3, base 4, etc.
2. What would 10, 100, 1000, 111, mean in these various bases?
3. Below we have begun addition tables in three bases. Guess the bases and complete the tables.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	0	2	3	4	5	.	.	.		
2	2	3	4	5						
3										
4		.								
5		.								
6		.								
7										
8										
9										

	0	1
0	0	1
1	1	

	0	1	2	3
0	0	1	2	3
1	1	2	3	10
2	2	.	.	.
3	.			

4. Indicate in which bases the following additions were performed:

$$\begin{array}{r}6\\3\\\hline 9\end{array},\quad \begin{array}{r}12\\3\\\hline 21\end{array},\quad \begin{array}{r}110\\11\\\hline 1001\end{array},\quad \begin{array}{r}13\\12\\\hline 31\end{array},\quad \begin{array}{r}111\\110\\\hline 1101\end{array}.$$

VIII. MULTI-BASE CALCULATOR

0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9
a	a	a	a	a	a
b	b	b	b	b	b

The preceding chart can be used as a calculator to perform addition and subtraction in different bases.

A. Addition

Let us illustrate the use of the calculator in adding the numbers 334 and 431 in base 5. First conceal all rows of numbers in the chart below the row of 4's; you can use a sheet of paper to do this. In figure 1 below, the dots indicate the concealed portion of the calculator, that part not needed for computation in base 5. We use the convention that the rows are ordered from right to left, so that column one is the first column on the right; column 2 is the second column from the right, etc.

Figure 1

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Figure 2

$$\begin{array}{cccccc} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

To begin the computation, the calculator is "zeroed," i.e., a poker chip or coin is placed over each zero in the chart. Figure 2 shows a zeroed machine. To put 334 into the machine, move the chip in the third column (from the right) down 3 spaces; move the chip in the second column down 3 spaces and move the chip in the first column down 4 spaces. The calculator now looks like figure 3.

Figure 3

$$\begin{array}{cccccc} \boxed{0} & \boxed{0} & \boxed{0} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & \boxed{3} & \boxed{3} & 3 \\ 4 & 4 & 4 & 4 & 4 & \boxed{4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Figure 4

$$\begin{array}{cccccc} \boxed{0} & \boxed{0} & 0 & 0 & 0 & \boxed{0} \\ 1 & 1 & \boxed{1} & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & \boxed{2} & 2 \\ 3 & 3 & 3 & \boxed{3} & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

To add in 431, we move the chip in column 3 down 4 additional spaces, starting from the position the chip already occupies in figure 3. After the chip reaches position 4 its next position is 0. Thus, the new position for the chip after "moving down" 4 spaces is position 2 in the third column.

Carrying Rule:

> Whenever we count to the end of a column and continue through 0 as we did in this case, the chip in the adjacent column to the left must be moved down one space. (Why?)

Similarly, we count down 3 additional spaces in the second column and 1 additional space in the first column. The calculator should now look like figure 4 (don't forget the carrying rule) and the sum is 1320.

1. Explain how this calculator could be used to add numbers in bases other than 5.
2. Use the calculator to add:

    4723 and 1654 base 8
    302 and 1331 base 5
    1624 and 769 base 10
    735 and 936 base a
    1658 and 936 base b.

3. Make up several other addition problems and use the calculator to solve them.
4. Can you now explain why this calculator works?

B. Subtraction

To see how subtraction is done, let us compute 1421 minus 342 in base 5. We first put 1421 into the machine as above; see figure 5. Now we "take away" 342 by counting up 3 spaces in the third column, 4 spaces in the second column and 2 spaces in the first column. After a chip reaches the 0 position, the next position is the 4 position.

Dropping Rule:

> Whenever we count to the top of a given column and continue through 4, the chip in the adjacent column to the left must be moved up one space. (Why?)

The final answer, 1024, is indicated in figure 6.

Figure 5

$\boxed{0}$	$\boxed{0}$	0	0	0	0
1	1	$\boxed{1}$	1	1	$\boxed{1}$
2	2	2	2	$\boxed{2}$	2
3	3	3	3	3	3
4	4	4	$\boxed{4}$	4	4
.	.	.	.	.	.
.	.	.	.	.	.

Figure 6

$\boxed{0}$	$\boxed{0}$	0	$\boxed{0}$	0	0
1	1	$\boxed{1}$	1	1	1
2	2	2	2	$\boxed{2}$	2
3	3	3	3	3	3
4	4	4	4	4	$\boxed{4}$
.	.	.	.	.	.
.	.	.	.	.	.

1. Explain how this calculator can be used to subtract numbers in bases other than 5.

2. Use the calculator to subtract:

   361 from 453 base 7
   222 from 301 base 4
   549 from 604 base 10
   789 from 801 base b

IX. AREAS WITH GEO-BOARDS

Select a geo board and a handful of rubber bands. A desirable board should have 1 inch spacing between pegs (small 3/4 nails) with a 13 x 13 array of the pegs (169 pegs) at the vertices of 1 inch squares covering the face of the board.

Let us call a square such as:

"one unit of area" and call each side of that square "one unit of length". With these definitions, calculate the area of several different rectangles and triangles found by hooking the rubber bands around the pegs. Also try to find the areas of other irregular figures.

1. Make up any closed figure and determine its area using only square units. If your figure is very irregular you might want to get:

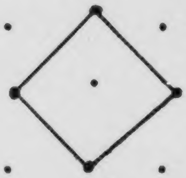
    a) A lower estimate of the area by counting the maximum number of squares which lie entirely within the figure. (side or a corner of a square might be par of the boundary.)

    b) An upper estimate of the area by adding to the lower estimate the number of squares which meet the boundary. (Do not add those squares with only a vertex or a side in common with the boundary.)

2. Form a 5 x 5 square by putting a large rubber band around such a square. How many squares of different areas can you find in this square?

    Could a square such as that pictured on the right be found in your 5 x 5 square?

    What is the area of this square?

    Although there are many ways to determine this area, the following procedure is convenient.

    Enclose the figure of unknown area as efficiently as possible inside a figure of known area. As illustrated above, it is enclosed within a square of area 4. (However, we also know the area of the four triangles which have been added, i.e., each has area 1/2.) Therefore, the area of our given square is 4 - 4 (1/2) = 2.

    Can you find a square with the areas of 5? 3? and 2 units?

3. Now, using the whole geo board, make up any simple closed polygon you like and determine its area. When this has been explored at some length, the following exercise will point out the importance of the number of pegs on the boundary and the inside.

4. Construct a figure on the geo board having the following properties and fill in the blanks. On graph paper copy the figures from the geo board.

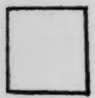
B	I	A
3	0	___
3	0	___
4	0	___
5	0	___
6	0	
3	1	___
4	1	___
5	1	___
6	1	___
3	2	___
4	2	___
5	2	___
6	2	___
	___	
___	___	___
___	___	___
___	___	___
___	___	___

B = Number of pegs on the boundary
I = Number of pegs in the interior
A = ARea of the polygon

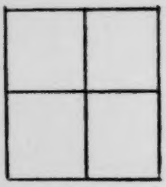
5. Can you find a relationship between B, I, and A based on the above data?

X. COUNTING SQUARES AND CUBES

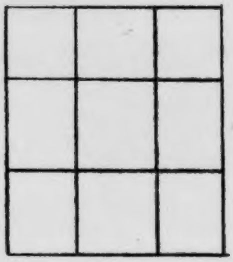
A. Consider these arrays of squares:



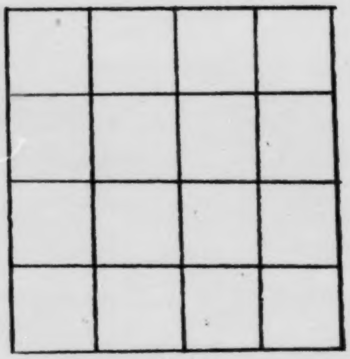
1x1



2x2



3x3



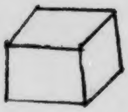
4x4

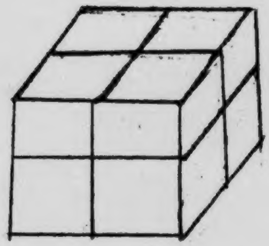
(1) Can you tell the number of rectangles (including squares) that you can see in each of the arrays?

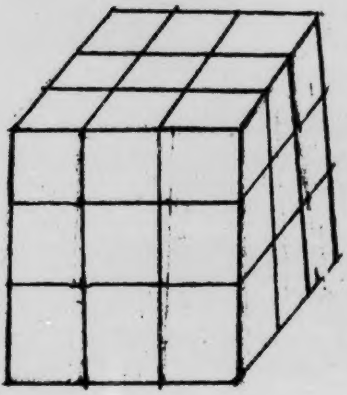
(2) Suppose you have a 5 x 5 array, can you predict the number of rectangles in this?

(3) Now predict the number of rectangles in a 7 x 7 array. What is the pattern? Can you give a formula for the number of rectangles in any nth array (an n x n square array of squares)?

B. Consider this series of arrays of cubes:







(1) Can you tell the number of different cubes that can be distinguished in each array?

(2) Suppose you have a 4 x 4 x 4 array, can you predict the number of cubes in it? Check your prediction and explain how you counted the number of cubes in the 4 x 4 x 4 array.

(3) Now predict the number of cubes in a 8 x 8 x 8 array. What is the pattern? Can you give a formula to describe this pattern?

XI. ARRAYS, POLYNOMIALS AND FINITE DIFFERENCES

Consider the following array:

10 3 16 73 198 415 748 1221

-7 13 57 125 217 333 473

20 44 68 92 116 140

24 24 24 24 24

1. Can you figure out how this table was constructed? Can you extend the array keeping the same pattern? Find and write the necessary numbers to extend it to the right.

2. Now, can you extend the array in the other direction such as to the left or down? Try it and put in some values. Can you extend the array upwards? Can you find the one hundredth number to the right of 10 in the first line?

3. These arrays are related to equations. For instance, if we were given the equation $y = 3x^2 + 5x + 2$ we would have the following array:

   2 10 24 44 70

   8 14 20 26

   6 6 6

   and if we had the equation $y = x^2 + 2x - 5$, we would have the following:

   -5 -2 3 10 19 30 43

   3 5 7 9 11 13

   2 2 2 2 2

   and if we had $y = 2x^2 + 7x + 13$, we would have:

   13 22 35 52 73 98

   9 13 17 21 25

   4 4 4 4

Can you predict the equation which has the following array?

1 9 27 55 93 141

8 18 28 38 48

10 10 10 10

The function for the above table is $y = 5x^2 + 3x + 1$. It was generated by tabulating values of y for x = 0, 1, 2, 3, 4, ... and taking differences between terms in one line to obtain terms in the next lower line. Thus, we have table for $y = 5x^2 + 3x + 1$:

X	0	1	2	3	4	5
Y	1	9	27	55	93	144

and the first row of the array is the sequence of y values.

4. Write down and explain the rules used to find the formula above.

5. Now, can you find an equation which would be similarly related to the array below?

A B C D E

P Q R S

W W W

All the formulas or equations we have discussed so far had an x squared term. Let us consider some different situations.

6. What would be the formula for y in terms of x related to the following array?

1 4 7 10 13

3 3 3 3

And for this array?

1 0 5 22 57 116

-1 5 17 35 59

6 12 18 24

6 6 6

The first array is related to $y = 3x + 1$ and the second to $y = x^3 - 2x + 1$.

The general analysis for formulating the value of a second degree function in one variable is suggested below. Additional data relative to first, third and fourth degree functions is also provided.

Given $y = ax^2 + bx + c$; this implies the following table:

X	Y	1st dif. in value of "Y"	2nd differences (dif. of dif.)
0	c		
		$a + b$	
1	$a + b + c$		$2a$
		$3a + b$	
2	$4a + 2b + c$		$2a$
		$5a + b$	
3	$9a + 3b + c$		$2a$
		$7a + b$	
4	$16a + 4b + c$		

The array derived from this would be:

$(c) \quad (a+b+c) \quad (4a+2b+c) \quad (9a+3b+c) \quad (16a+4b+c)$

$(a+b) \quad (3a+b) \quad (5a+b) \quad (7a+b)$

$(2a) \quad (2a) \quad (2a)$

Let us stop our analysis now, and consider the inductive situation which we often face in practice. We will attempt to reconstruct the original equation. We note that:

(a) We needed 2nd differences before obtaining unchanging difference terms. This implies a quadratic.

(b) The coefficient of the squared term is 1/2 the value of the 2nd difference (which, as indicated, is constant).

(c) The constant in the quadratic expression is the first term of the first line in the array.

(d) The coefficient of the linear term in the expression is obtained by the subtracting the quadratic coefficient (represented originally by "a") from the first term in the second line (which our analysis showed to equal "a+b").

Similar analyses can be made based upon the following analyses.

1. $y = ax + b$

x	y	
0	b	
		a
1	$a + b$	
		a
2	$2a + b$	
		a
3	$3a + b$	

one difference

first degree

2. $y = ax^2 + bx + c$

x	y		
0	c		
		$a + b$	
1	$a + b + c$		$2a$
		$3a + b$	
2	$4a + 2b + c$		$2a$
		$5a + b$	
3	$9a + 3b + c$		$2a$
		$7a + b$	
4	$16a + 4b + c$		

two differences

second degree

3. $y = ax^3 + bx^2 + cx + d$

x	y			
0	d			
		$a + b + c$		
1	$a + b + c + d$		$6a + 2b$	
		$7a + 3b + c$		$6a$
2	$8a + 4b + 2c + d$		$12a + 2b$	
		$19a + 5b + c$		$6a$
3	$27a + 9b + 3c + d$		$18a + 2b$	
		$37a + 7b + c$		$6a$
4	$64 + 16b + 4c + d$		$24a + 2b$	
		$61a + 9b + c$		
5	$125a + 25b + 5c + d$			

three differences

third degree

4. $y = ax^4 + bx^3 + cx^2 + dx + c$

x	y				
0	e				
		a + b + c + d			
1	a + b + c + d + e		14a + 6b + 2c		
		15a + 7b + 3c + d		36a + 6b	
2	16a + 8b + 4c + 2d + e		50a + 12b + 2c		24a
		65a + 19b + 5c + d		60a + 6b	
3	81a + 27b + 9c + 3d + e		110a + 18b + 2c		24a
		175a + 37b + 7c + d		84a + 6b	
4	256a + 64b + 25c + 5d + e		194a + 24b + 2c		
		369a + 61b + 9c + d			
5	625a + 125b + 25c + 5d + e				

Exercises

1. Find a formula that relates y to x in the following tables:

(a)

x	y
0	1
1	4
2	7
3	10
4	13

(b)

x	y
0	0
1	2
2	8
3	18
4	32

(c)

x	y
0	0
1	2
2	6
3	12
4	20
5	30

2. Find a formula that relates y to x in the following tables:

(a)

x	y
0	b
1	1 + b
2	2 + b
3	3 + b
4	4 + b

(b)

x	y
0	0
1	2 + b
2	8 + 2b
3	18 + 3b
4	32 + 4b
5	50 + 5b

(c)

x	y
0	0
1	a
2	2a
3	3a
4	4a
5	5a

3. Graph the following points, (0, -45), (1, 0), (2, 21), (3, 24), (4, 15).

   (a) What kind of curve do these points form?

   (b) Find a formula for the curve.

4. One number in each of these two sequences is misprinted. Determine which one and explain.

   (a) 1, 2, 4, 8, 15, 27, 42, 64, 93

   (b) 1, 3, 11, 31, 69, 113, 223, 351, 521, 739, 1011

5. A table of values showing the number of terms in a sum and the value of the sum can be developed to find a formula for the value of a sum, given the number of terms in the sum. Then a formula generalizing the pattern of the table can be developed using the method of finite differences. Find the sum of the first 93 odd counting numbers using this method.

6. Developing a table as suggested in (5), find a formula for the sum of the n odd counting numbers that follow the p-th odd counting number.

- SETS AND LOGIC -

I. Sets

Exercises

1. Use a box of attribute blocks to form the following sets:

   a) The set of all blocks that are blue and large.

   b) The set of all blocks that are not square and large.

   c) The set of all blocks that are not square and large.

   d) The set of all blocks that are diamonds or small.

   e) The set of all blocks that are red and not circular.

   f) The set of all blocks that are not diamonds and not large.

   g) The set of all blocks that are blue or large and not square.

   h) The set of all blocks whose surface area is the same on all sides.

   i) The set of all blocks that are not yellow, and six sided.

   j) The set of all blocks that are green and three sided.

   k) The set of all blocks that are green and three sided.

2. Use loops from the attribute set to do the following:

   (1) Place a green loop inside a black loop.

   (2) Place a small red triangle and a small red square inside the green loop.

   (3) Place a large green triangle and a large green square inside the black loop but not in the green loop.

Questions:

(1) List the members of the set of things inside the green loop.

(2) List the members of the set of things inside the black loop.

(3) Are all the members of the set of things inside the green loop also members of the set of things inside the black loop? Explain.

(4) What name can be applied to the set of things inside the green loop in connection with the set of things in the black loop?

(5) Move the green loop so that part of it projects outside the black loop with the red objects still within the black loop. What relationship exists?

(6) What name is given to the set of triangles with respect to the set of objects within the black loop?

(7) Give the members of another subset of objects within the black loop.

(8) Is the set of green objects a subset of the set of objects within the black loop?

3. Use the attribute sheet to help determine the number of figures that have the attribute specified in each of the following statements, and make any general statement that is supported by your observation. How many figures are in

____1. hatched set? ____1. non-hatched set?

____2. solid set? ____2. non-solid set?

____3. square set? ____3. non-square set?

____4. triangle set? ____4. non-triangle set?

____5. large set? ____5. non-large set?

____6. small set? ____6. non-small set?

____

____

____1. hatched and square set? ____1. hatched or square set?

____2. non-hatched and circle set?____2. non-hatched or circle set?

____3. hatched and small set? ____3. hatched or small set?

____4. non-hatched and large set? ____4. non-hatched or large set?

____5. circle and small set? ____5. circle or small set?

____6. triangle and large set? ____6. triangle or large set?

____

____

____1. square and not square set? ____1. square or not square set?

____2. hatched and not hatched set?____2. hatched or not hatched set?

____3. small and not small set? ____3. small or not small set?

____4. square and circle set? ____4. square or circle set?

____5. small and large set? ____5. small or large set?

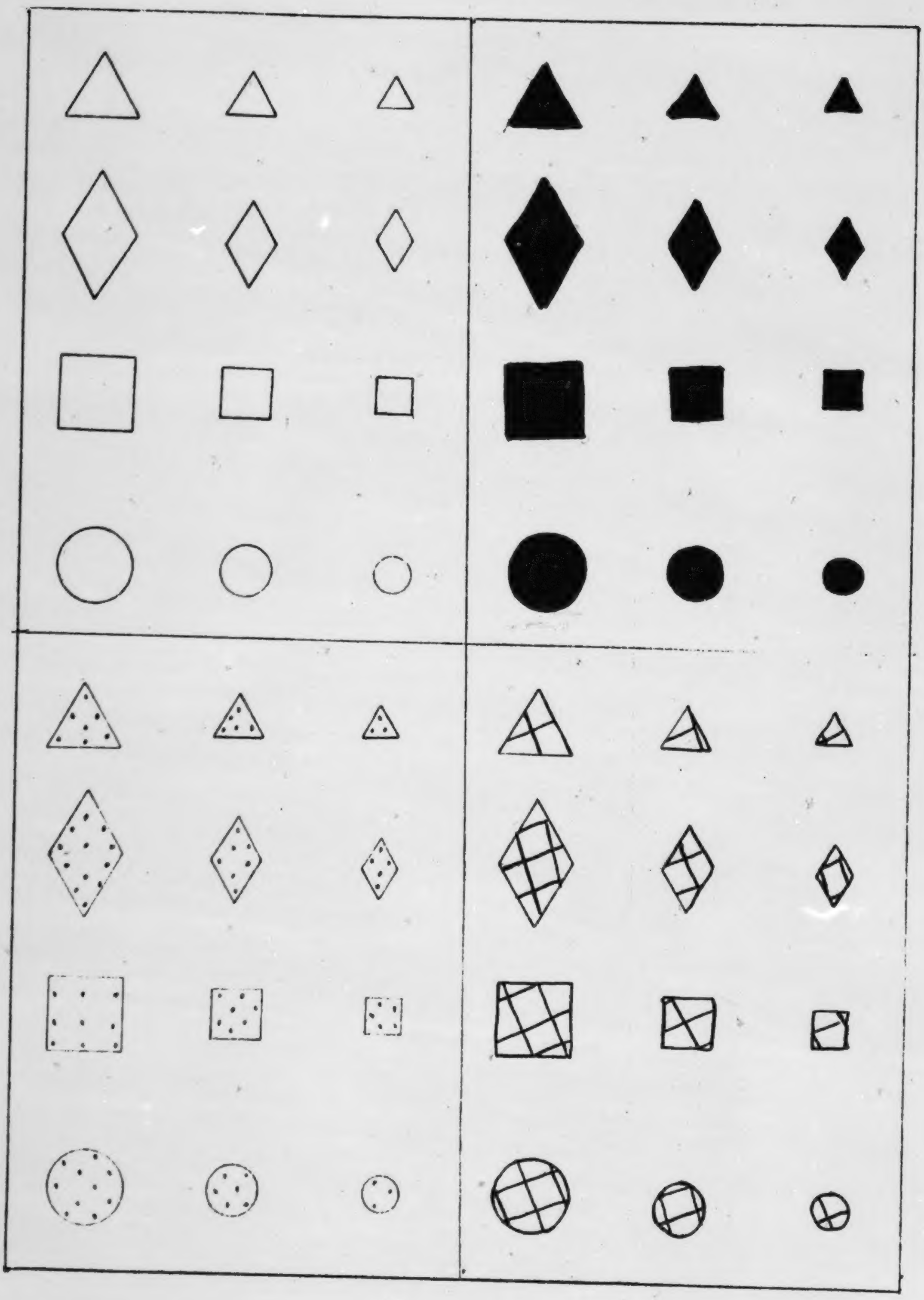
____6. small and circle set? ____6. small or circle set?

____

____

_____1. How many in square set?	_____1. the not not square set?
_____2. small or hatched set?	_____2. hatched or small set?
_____3. large and circle set?	_____3. circle and large set?
_____4. not "square and small"set?	_____4. not square or not small set?
_____5. hatched and circle set?	_____5. circle and hatched set?
_____6. triangle or hatched set?	_____6. hatched or triangle set?

ATTRIBUTE SHEET



4. Use the sets of attribute figures on the next page to determine your answers.

a. Make a check (✓) in the parenthesis by each relation that expresses the way the set of attribute figures in the dash-circle is related to the set of attribute figures in the line-circle.

1. ( ) Equal ( ) Equivalent ( ) Subset
2. ( ) Equal ( ) Equivalent ( ) Subset
3. ( ) Equal ( ) Equivalent ( ) Subset
4. ( ) Equal ( ) Equivalent ( ) Subset
5. ( ) Equal ( ) Equivalent ( ) Subset

How should we define these relations?

b. Make a check in the parenthesis by the answer that is the set of attribute figures in the line-circle union the set of attribute figures in the dash-circle.

1. ( ) Line-Circle ( ) Dash-Circle ( ) Empty Set ( ) None of These
2. ( ) Line-Circle ( ) Dash-Circle ( ) Empty Set ( ) None of These
3. ( ) Line-Circle ( ) Dash-Circle ( ) Empty Set ( ) Dot-Circle
4. ( ) Line-Circle ( ) Dash-Circle ( ) Empty Set ( ) None of These
5. ( ) Line-Circle ( ) Dash-Circle ( ) None of These ( ) Dot-Circle

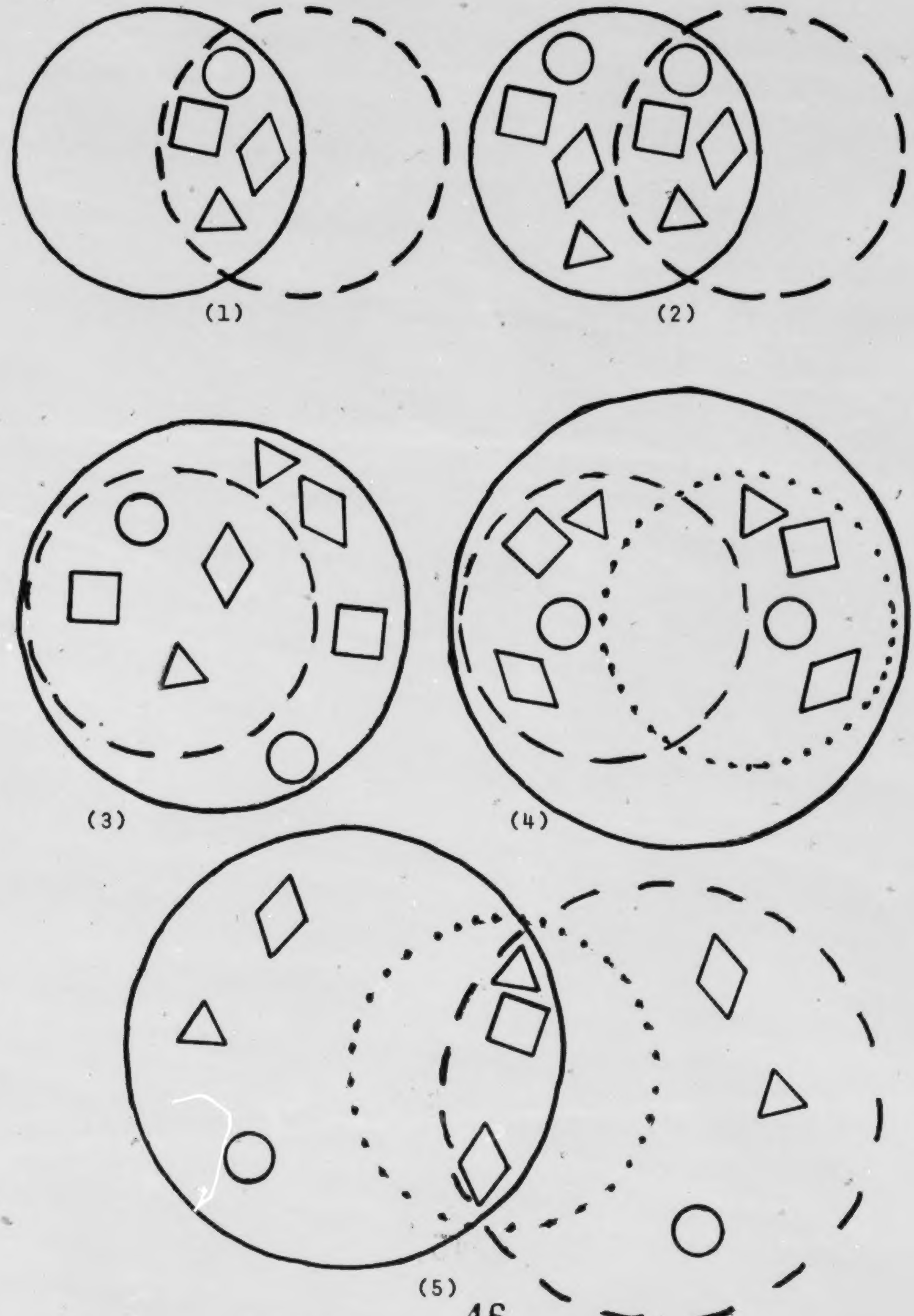
How should we define this operation?

c. Make a check in the parenthesis by the answer that is the set of attribute figures in the line-circle intersect the set of attribute figures in the dash-circle.

1. ( ) Line-Circle ( ) Dash-Circle ( ) Empty Set ( ) None of These
2. ( ) Line-Circle ( ) Dash-Circle ( ) Empty Set ( ) None of These
3. ( ) Line-Circle ( ) Dash-Circle ( ) Empty Set ( ) Dot-Circle
4. ( ) Line-Circle ( ) Dash-Circle ( ) Empty Set ( ) None of These
5. ( ) Line-Circle ( ) Dash-Circle ( ) None of These ( ) Dot-Circle

How should we define this operation?

SETS OF ATTRIBUTE FIGURES



(1) (2)

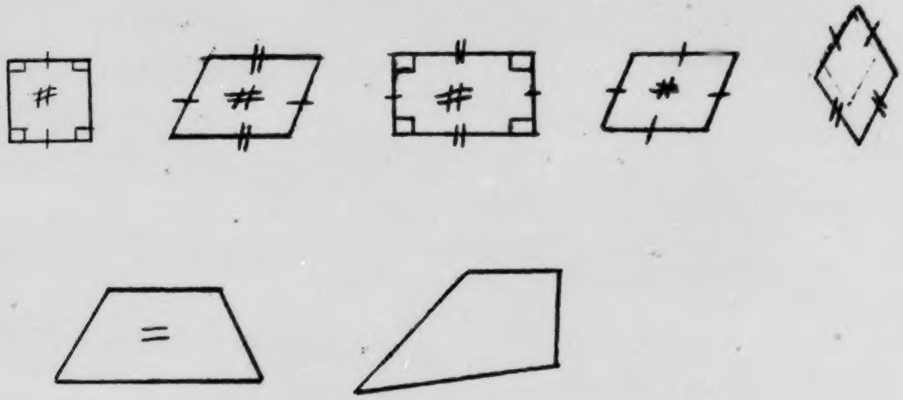
(3) (4)

(5)

5. If "=" appears in the interior of a quadrilateral, the opposite sides going in the same direction are parallel.

Sides or angles with the same mark have the same length or measure respectively.

Make a table for the intersection of the following set of quadrilateral.



II. SUBSETS

1. Suppose we have a set containing no elements, i.e. the empty set. How many subsets can be formed which contain no elements?

2. Given a set containing one element, how many subsets have no elements? 1 element?

3. Given a set containing 2 elements, how many subsets have no element? 1 element? 2 elements? etc.

number of elements in the set	number of subsets having exactly the following number of elements:											
	0	1	2	3	4	5	6	7	...	i	...	n
0												
1												
2												
3												
.												
.												
.												
n												

(a) What do you notice about the array of numbers in the table?

(b) How can the total number of subsets of any given set be determined?

(c) Suppose a given line in the array is:

1 9 36 84 126 126 84 36 9 1

What is the next line in the table?

(d) Given line: 1 11 55 165 330 462 462 330 165 55 11 1
then the next line is:

(e) What is the total number of subsets with a given number of elements i for a set with 3 elements? 4 elements? 5 elements? 6 elements? n elements?

Let's record some of this data in the above table.

Exercises

1. Consider the set A = {b,c} and set B = {a}. Putting the members of both sets A and B into a new set, one gets the set which can be called C, that is, C = {a,b,c}.

   Using the above example, do the following:

   (1) If A = {1, 2, 3} and B = {4, 5}, then C ={ }.

   (2) If A = {2, 3, 4} and B = {4, 5}, then C ={ }.

   (3) If A = {x, y, z} and B = {w, x, y}, then C ={ }.

   Questions:

   (1) In each case, what do you notice about each element in set C with regard to sets A and B?

   (2) Are there any repeated or duplicated elements in set C?

   (3) What name is given to set C?

2. Suppose set A = {a,b,c} and its subsets are named as B = {a}, C = {b}, D = {c}, E = {a,b} , F = {a,c}, G = {b,c}, H = {a,b,c}, and I = 0. Fill in the following tables for ∩. After completing the table, study it closely and make as many observations as you can concerning the intersection of the sets involved.

∩	B	C	D	E	F	G	H	I	
B									
C									
D									
E									
F									
G									
H									
I									

OBSERVATIONS:

What are some properties of this operation?

3. Complete the following table for (union) using the same sets as given above.

$\cup$	B	C	D	E	F	G	H	I	
B									
C									
D									
E									
F									
G									
H									
I									

OBSERVATIONS:

4. Construct tables for union and intersection using the following collection of subsets:

   a. ϕ, [1, 2, 3], [1], [1, 2]

   b. ϕ, [1, 2, 3], [1]

   c. ϕ, [1, 2, 3], [1], [2]

   Compare the properties of the operations on these subsets.

III. NEW SETS FROM OLD

Let us continue our discussions of sets by playing this game. Here are the rules:

1. Class is divided into two teams.
2. Four coins are to be used in this game (a penny, a nickel, a dime, and a quarter). Let's call the set of coins A = [p, n, d, q]. Ask the students to list the subsets of A: [], [p], [n], [d], [q], [p, n], [p, d], [p, q], [n, d], [n, q], [d, q], [p, d, n], [p, d, q], [p, n, q], [n, d, q], [p, n, d, q].
3. List all of the amounts that can be formed from the coins above.
4. The object of the game is to accumulate the most money.
   (a) Each play must be followed by a play which must have one or more coins in common with the previous play.
   (b) A play may not be used more than once during a game. The penalty for repeating a play might be the loss of a turn or adding the amount to the opposing team. The game is over only when all plays have been used or a player surrenders.

   Example: 35¢ may be followed by 15¢ because they have a dime in common. After a winner has been declared, the class should examine the data. The instructor should now give a formal definition for intersection.

Exercises

1. If A is the set of all even whole numbers and B is the set of all odd numbers, what is the set $A \cup B$? What is $A \cap B$?

2. Let A and B be sets. Give conditions under which:

   (a) $A \cup B = A$

   (b) $A \cap B = A$

   (c) $A \cup \emptyset = A$

   (d) $A \cap \emptyset = A$

   (e) $A \cap B = \emptyset$

   (f) $A \cup B = \emptyset$

   (g) $A \cup B = B \cup A$

   (h) $A \cup B = A \cap B$

3. If $R = \{x \mid x \in I \text{ and } x + 11 \geq 20\}$ and $S = \{y \mid y \in I \text{ and } y - 11 < 101\}$ find $R \cup S$ and $R \cap S$.

4. Is it possible for $X \cup Y$ to the the same set as X? Explain.

5. Is $S \cup T$ the same set as $T \cup S$ for all sets S and T?

6. Using only intersection and union, you can work this puzzle. Each resulting set of letters will need to be unscrambled to make a word. The four words read in order will make a familiar statement.

Across:

1. $A = \{a,h,G,s,d,o,p,o\}$. $A \cap B =$ ____________
   $B = \{r,d,f,o,G,l,o\}$.
2. $C = \{f,e,g\}$. $C \cup D =$ ____________
   $D = \{i,r\}$.
3. $E = \{h,i,s,e,m,r,a,q,C,l\}$. $E \cap F =$ ____________
   $F = \{w,C,n,l,x,h,r,e,i,p,a\}$
4. $G = \{n,B,w\}$. $G \cup H =$ ____________
   $H = \{r,o\}$

▨	▨	1.				▨	▨	▨
▨	▨	▨	▨	2.				
3.							▨	▨
▨	▨	4.					▨	▨

IV. LOGIC

Exercises

1. Construct addition and multiplication tables for modular two system. Then replace T and F in the truth tables with 1 and 0 respectively. Consider the following tables:

p	q	p + q	q + p
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

"OR" Table

p	q	p . q	q . p
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

"AND" Table

Hence, the operations "OR" and "AND" are commutative. Note that 1 + 1 = 1. In the decimal system of real numbers 1 + 1 ≠ 1. Thus the students need to realize that they are working in a different system.

Also, consider the following tables:

p	q	r	(p+q)	(p+q)+r	(q+r)	p+(q+r)	(p.q)	(p.q).r	(q.r)	p.(q.r)
0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	0	0	0	0
0	1	0	1	1	1	1	0	0	0	0
0	1	1	1	1	1	1	0	0	1	0
1	0	0	1	1	0	1	0	0	0	0
1	0	1	1	1	1	1	0	0	0	0
1	1	0	1	1	1	1	1	0	0	0
1	1	1	1	1	1	1	1	1	1	1

From this table, we see that the associative laws hold for the operations "OR" and "AND".

Now, consider the following tables:

p	q	r	q+r	p.(q+r)	p.q	p.r	(p.q + p.r)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

From this table, we see that

$$p.(q+r) = p.q + p.r.$$

p	q	r	(q.r)	p+(q.r)	(p+q)	(p+r)	(p+q) . (p+r)
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Notice that $p+(q.r) = (p+q) . (p+r)$. However, in the decimal (real number) system $p+(q.r) \neq (p+q) . (p+r)$. This is another point of difference.

Again, consider:

p	q	(p+q)	~(p+q)	~p	~q	(~p) . (~q)
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

This table shows that ~(p+q) = (~p) . (~q) which is not true in the decimal system.

Consider the following table:

p	q	p.q	~(p.q)	~p	~q	(~p) + (~q)
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

This table shows that ~(p.q) = (~p) + (~q), which is not true in the decimal system.

These two results:

~(p+q) = (~p) . (~q) and

~(p.q) = (~p) + (~q)

are known as De Morgan's Laws. These two relationships are referred to as "The Duality Principle".

Compare other properties of this system with the modular two system.

2. The following abbreviations are used to describe the attributes of the various blocks.

⊙ : This is a circle.

△ : This is a triangle.

□ : This is a square.

⬡ : This is a hexagon.

◇ : This is a rhombus.

l : This is a large block.

s : This is a small block.

g : This is a green block.

b : This is a blue block.

y : This is a yellow block.

Note: Each of the sentences above is an atomic sentence.

(1) Give a verbal translation of l.

(2) As it relates to the blocks in your collection, can you say that l is the negation of s? Explain.

(3) Choose a set where l is the negation of g.

(4) Is "□" the negation of "□"? Justify your answer.

(5) Consider the atomic statement:

p: Some blocks are circles.

Which, if any, of the following statements correctly represents p? Justify your selection(s).

a) All blocks are circles.

b) Some blocks are not circles.

c) It is not true that some blocks are circles.

d) All blocks are not circles.

e) No blocks are circles.

f) Not all blocks are not circles.

If more than one of the statements were suitable, is either of them more concise in stating ~p than the others? Explain.

(6) Again consider the statement

p: Some blocks are circles.

a) What is meant by ~~p?

b) Is it the same as ~(~p)? Explain.

c) Assuming that p is true, is ~(~p) true or false? Explain.

d) Assuming that p is false, is ~(~p) true or false?

e) Make a general statement comparing the truth value of a statement with that of the negation of its negation.

(7) Which blocks are described by (s∧b)∧ ⊙ ?

(8) Which ones are described by s∧(b∧ ⊙)?

(9) Compare the verbal form of (s∧b)∧ ⊙ with that of s∧(b ∧ ⊙).

(10) How would you expect the translation of (□ ∧ 1)∧g to compare with that of □ ∧(1∧g)? Verify your guess by listing the blocks described by each statement.

(11) Make a general statement comparing p∧(q∧g) and (p∧q) ∧ g, when p, q and g are atomic sentences.

(12) Which blocks are described by (⬡ ∧ y)? (y ∧ ⬡)?

(13) Which blocks are described by (□ ∧ 1)? (1 ∧ □)?

(14) Suppose W and U are any two atomic sentences, would (W∧U) mean the same as (U∧W)? Explain.

(15) Determine which blocks are described by [(⊙ ∧ ~b) (1 ∧ ~g)] ∧ (~y∧g). Verify your choice by constructing a truth table.

(16) Which blocks are described by (△ ∨ 1) ∨ y?

(17) Which blocks are described by $\triangle \vee (1 \vee y)$?

(18) How would you expect the blocks described by $g \vee (\square \vee \odot)$ to compare with those described by $(g \vee \square) \vee \odot$? Verify your guess.

(19) Make a general statement comparing $p \vee (q \vee g)$ and $(p \vee q) \vee g$, where p, q and r are atomic sentences.

(20) Suppose p, q and r are atomic sentences. How do you expect $p \underline{\vee} (q \underline{\vee} g)$ to compare with $(p \underline{\vee} q) \underline{\vee} g$? Verify your guess by selecting three of the atomic sentences listed at the beginning of the worksheet.

(21) Investigate the comparison of $p \vee q$ with that of $q \vee p$ where p and q are atomic sentences. Do the same for $p \underline{\vee} q$ and $q \underline{\vee} p$.

(22) Determine which blocks are described by $[(y \wedge {\sim}\triangle) \vee ((\triangle \underline{\vee} 1) \wedge {\sim}(\triangle \vee s)]$ $(\bigcirc \underline{\vee} s)$. Verify your answer.

(23) How many lines would a truth table for $b \underline{\vee} (g \underline{\vee} \square)$ contain? Explain.

(24) Make a general statement comparing the number of lines in a truth table for K different atomic sentences with the number of lines in a truth table for $(K + 1)$ different atomic sentences.

(25) Make a truth table for $b \vee (g \wedge \square)$. What was the last column completed?

(26) Make a truth table for $(b \vee g) \wedge (b \vee \square)$. What was the last column completed?

(27) Compare the truth values of the molecular statement in #25 with that in #26 (when the truth values of the atomic sentences are the same). Make a general statement.

(28) Compare the truth tables of $b \wedge (g \vee \square)$ with that of $(b \wedge g) \vee (b \wedge \square)$.

(29) By constructing truth tables, determine whether $b \wedge (g \vee \square)$ is the same as $(b \wedge \square) \wedge (b \vee g)$.

(30) Which blocks are described by $(\triangle \vee ({\sim}\triangle))$?

(31) Make a truth table for $(\triangle \vee ({\sim}\triangle))$.

(32) A statement whose truth values are the same as those shown in the table for $(\triangle \vee ({\sim}\triangle))$ is called a tautology. Give a definition for a tautology.

(33) Statements with always the same truth values are called equivalent statements. Make truth tables for each of the statements below. Then:

a) Find all equivalent sentences.

b) Find all forms of negation of statements.

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge \sim q$	$q \leftrightarrow p$	$p \leftrightarrow \sim q$	$\sim p \leftrightarrow \sim q$
t	t							
t	t							
f	t							
f	f							
$\sim(\sim p)$	$\sim(\sim q)$	$p \vee q$	$p \wedge q$	$\sim p \wedge q$	$\sim p \wedge \sim q$	$\sim p \leftrightarrow q$	$\sim(\sim p \vee \sim q)$	$(\sim p \vee q) \wedge (p \wedge \sim q)$
$p \rightarrow q$	$p \leftrightarrow q$	$\sim(p \vee q)$	$\sim(p \wedge q)$	$p \rightarrow \sim q$	$\sim p \rightarrow q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$	$[q \wedge (p \rightarrow q)] \rightarrow p$
							t t t t	
$\sim p \rightarrow q$	$\sim(p \leftrightarrow q)$	$(p \vee \sim q)$	$\sim p \vee q$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$	$\sim(p \wedge \sim p)$	$(p \rightarrow q) \wedge (\sim q \rightarrow \sim p)$	$p \vee \sim(p \wedge q)$

The lasts two columns of the above table are quite different from the preceding ones. They require information not found in the table. The final column had all T's except for one f. The other column had all t's. These columns are forms of arguments in logic. Arguments in logic are valid or invalid according to their form (irrespective of what may be true or false in reality).

The last three formulas on the bottom now have a striking property; they are true regardless of the truth value of the atomic sentences. Such a sentence is called a tautology. Ask students to find other tautologies in this table.

V. SWITCHES AND BATTERIES

Exercises

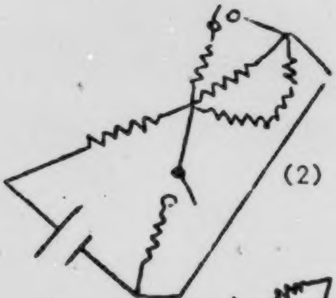
1. Listed below are the permanent members of the United Nations Security Council (China, U.S.A., U.S.S.R., England, France)

   a) Assuming only U.S.A. and U.S.S.R. have veto power and if a resolution is to pass a majority vote is needed. Draw a schematic diagram using batteries, switches, and resistors (to represent the members) to show what could possibly happen.

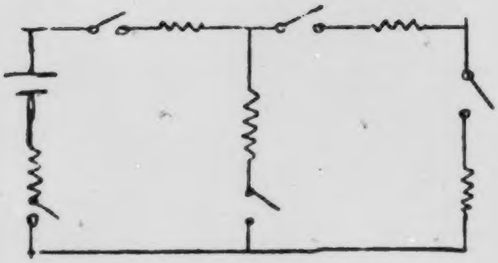
   b) Let us assume all five members have veto power. Draw a diagram to show when the resolution has to be passed.

   c) Draw a diagram when no one has veto power.

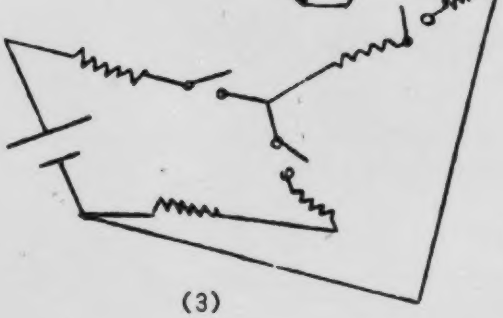
   d) Simplify and give meaning of the following circuits.



(2)



(1)



(3)

2. Suppose there are only two members of the Security Council (U.S.A., U.S.S.R.) both have to say yes if the resolution is passed.

   a) Draw a diagram and make a table.

   b) If we let yes be 1, and no be zero, then what will the table look like?

3. Repeat problem 2 with only one of them having to vote yes if the resolution has to be passed.

4. Let's see what happens if we think of a series circuit as being similar to multiplication and a parallel circuit as being equivalent to addition but with a funny rule for adding 1 and 1.

   Let's use (+) and (x) as the addition and multiplication symbols in this algebra. Make a table or tables to show a correspondence between switching circuits and addition and multiplication.

   Write the formal properties of real numbers.

VI. LOGICAL ARGUMENTS

Exercise - Determine for each of the following arguments whether it is valid or invalid.

1. A set of premises with a proposed conclusion is called an argument. If the conclusion follows logically from the premises, the argument is said to be valid; otherwise it is invalid. Determine for each of the following arguments whether it is valid or invalid.

(1) If prices are going still higher, inflation is surely coming.
Prices are going still higher.
∴ Inflation is surely coming.

(2) If it is raining, I will wear my raincoat.
It is raining.
∴ I will wear my raincoat.

(3) If he is tall, he is thin.
He is thin.
∴ He is tall.

(4) If he is a sailor then he goes to sea.
He does not go to sea.
∴ He is not a sailor.

(5) If today is Thursday, then yesterday was not Tuesday.
Yesterday was Wednesday.
∴ Today is Thursday.

(6) If it is green, then it is grass.
It is green.
∴ It is grass.

(7) If Los Angeles is not in California, then golf balls are not sold in Chicago.
Golf balls are sold in Chicago.
∴ Los Angeles is in California.

(8) If the week has seven days, then fish have wings.
If fish have wings then there are fifty-two weeks in a year.
∴ If the week has seven days, then there are fifty-two weeks in a year.

(9) If he gambles, then he loses money.
He does not gamble.
∴ He does not lose money.

(10) The given triangles are similar.

∴ The triangles are mutually equiangular.

If the given triangles are mutually equiangular, then they are similar.

(11) If angle A = angle B, then AC = BC.

AC = BC

∴ Angle A = Angle B

(12) If it is a wab, then it is a yud.

It is a wab.

∴ It is a yud.

(13) It is round, if it is blue.

If it is blue, then it holds water.

∴ If it is round, then it holds water.

(14) If Delaware is larger in area than Texas, then Australia is larger than the U.S.

Delaware is larger in area than Texas.

∴ Australia is larger than the U. S.

(15) The days are becoming shorter.

"The nights are becoming longer" implies that "the days are becoming shorter".

∴ The nights are becoming longer.

(16) If X passes the final examination, then he will pass the course.

X will pass the course.

∴ X will pass the final examination.

(17) If wages are not raised, then buying decreases.

If there is a depression, wages cannot be raised.

∴ If there is a depression, then buying cannot increase.

(18) If my father gives me a dollar, then I'll get a milkshake.

∴ My father gave me a dollar.

I get a milkshake.

(19) If it is a Ford, it burns oil.

It is not a Ford.

∴ It does not burn oil.

Having studied these arguments intuitively, we are ready for a formal approach to *validity* and *invalidity*.

First, an argument should be translated into symbolic notation. This in effect analyzes the argument into its logical structure and suppresses all referential content.

Second, a truth table is prepared for the premises and the conclusion, and it is examined for proof of validity or invalidity applying the following definition:

> An argument is valid if and only if, on any row of the truth table where the premises are *all* true, the conclusion is also true, i.e., *there* is no row on which the premises are all true and the conclusion false.

Thus, an argument is *invalid* if there is any one line with true premises and a false conclusion.

2. Translate each of the statements above into symbolic notation. Prepare truth tables for both the premises and the conclusion. Examine the truth table for proof of validity or invalidity according to the definition for a valid argument as given in class.

3. Translate each of the following composite sentences in symbolic language, using letters to stand for the prime components.

   a) Either it is raining or someone left the shower on.

   b) A prime number is either the integer 2 or it is an odd integer.

   c) If it is foggy tonight, then either John must stay home or he must take a taxi.

   d) If it rains, then it pou-s.

   e) John will sit, and he or George will wait.

   f) A triangle is equilateral if and only if it is equiangular.

   g) John will sit and wait, or George will wait.

   h) I will go either by bus or by taxi.

   i) The course is difficult and the grades are high.

   j) Neither the north nor the south won the civil war.

   k) It is not true that the course is difficult.

   l) If, and only if, irrigation ditches are dug will the crops survive; but if the crops do not survive, then the farmer will go bankrupt and leave.

   m) Early to bed and early to rise makes a student alert in class.

   n) If I am either tired or hungry, then I cannot study.

   o) If John goes to school, he will be happy, and if he does not go to school, he will not be happy.

4. Let C be "Today is clear."
   R be "It is raining today."
   S be "It is snowing today."
   Y be "Yesterday was cloudy."

Translate each of the following into acceptable English.

a) $C \rightarrow \sim R \wedge Y$

b) $Y \leftrightarrow C$

c) $\sim C \vee S$

d) $\sim R \leftrightarrow Y$

e) $Y \wedge (C \vee R)$

f) $(Y \rightarrow R) \vee C$

g) $C \leftrightarrow (R \wedge \sim S) \vee Y$

h) $(C \leftrightarrow R) \wedge (\sim S \vee Y)$

i) $\sim R \wedge \sim S$

j) $\sim(\sim R \vee S) \rightarrow C$

5. (1) Construct a truth table for each of the following:

a) $\sim p \vee (p \wedge q)$

b) $p \wedge (p \rightarrow q)$

c) $(\sim p \vee \sim q) \wedge r$

d) $\sim p \rightarrow q$

e) $p \wedge (q \ r) \leftrightarrow (p \wedge q) \wedge r$

(2) By use of the truth tables determine which of the following are tautologies and which are not tautologies.

a) $(p \wedge q) \rightarrow p$

b) $(p \vee q) \rightarrow p$

c) $(\sim p \ \sim p) \leftrightarrow p$

d) $(p \wedge q) \leftrightarrow (q \wedge p)$

e) $[p \rightarrow (p \ r)] \leftrightarrow [(p \rightarrow q) \ (p \rightarrow r)]$

f) $\sim(p \vee \sim p)$

6. Examine the validity of each argument.

(1) a) If the rod is spared, the child is spoiled.
b) The child is spoiled.
Conclusion: The rod has been spared.

(2) a) If Mr. X is registered to vote, he is at least 18 years old.
b) Mr. X is not registered to vote.
Conclusion: Mr. X is not yet 18 years old.

(3) a) If girls are pretty, they are popular with the boys.
b) Smart girls are not popular with the boys.
Conclusion: Pretty girls are not smart.

(4) a) When it rains, I get chilled.
b) Only when I get chilled will I get sick.
Conclusion: I got sick, therefore it rained.

(5) a) If the rod is spared, the child is spoiled.
b) The child is not spoiled.
Conclusion: The rod was not spared.

7. By using DeMorgan's law

(1) Write an equivalent expression for each of the following:

a) $\sim(\sim p \vee \sim q)$ b) $\sim(p \rightarrow q) \wedge (q \rightarrow r)$

c) The course is not difficult and the grades are not low.

d) It is not the case that I either go by bus or by taxi.

(2) Verify by the method of truth tables that the following sentences are true.

a) $[\sim(p \wedge q)] \leftrightarrow [(\sim p) \vee (\sim q)]$

b) $[\sim(p \vee q)] \leftrightarrow [(\sim p) \wedge (\sim q)]$

Suggested Items for Self Evaluation

A statement is a declarative sentence which can be judged true or false but not both.

(1) Which of the following are statements? Give the truth value of each statement. Mark "X" for sentences which are not statements.

a) ______How old are you?

b) ______The sun is shining.

c) ______It is raining.

d) ______What is your name?

e) ______The moon is made of cheese.

f) ______Students be quiet!

g) ______Two is an even integer.

h) ______The sum of the interior angles of a triangle is 180°.

i) ______I am hungry.

j) ______December is the first month of the year.

(2) Connectives play an important role in mathematics and mathematical logic.

a) List both the verbal and symbolic forms of each connective.

In b) and c) underline the word in parentheses which makes the sentence true.

b) A statement which contains a connective is a(n) (atomic/molecular) statement.

c) A statement which does not contain a connective is a(n) (atomic/molecular) statement.

d) Indicate which of the following are molecular statements.

(1) The sky is blue. __________

(2) The sky is not blue. __________

(3) If the sky is blue, then it is raining. __________

(4) Are you driving or are you walking? __________

(3) Define the terms valid and invalid.

Choose letters to represent the various atomic sentences and convert the statements in the next three exercises to symbolic form.

(4) If all students make good grades and mathematics is not puzzling, then either the day is hazy or the road is rough.

(5) If I lead a charmed life, then I am a good student; or if I do not lead a charmed life, then either I must study hard or I will be on probation.

(6) I will pass and I work hard if I do not work hard and I do not pass.

(7) Below is part of one line from a truth table. Complete the line.

$$[(p \leftrightarrow r) \vee (s \rightarrow p)] \wedge (p \acute{\vee} q)$$

Consider the statement, "If Mary has red hair, then Jim has brown eyes." Write in words and symbols the following:

(8) The inverse of the statement.

(9) The converse of the statement.

(10) The contrapositive of the statement.

(11) Which of the statements 8-10 are not always equivalent to (the same as) the original statement?

(12) Determine if the statement $p \leftrightarrow {\sim}{\sim}p$ is a tautology.

(13) Construct a truth table for the statement:

$(q \rightarrow r) \leftrightarrow [({\sim}s \wedge p) \wedge (s \vee q)]$

Determine whether the following arguments are valid or invalid.

(14) If X passes the final examination, then he will pass the course.

X will pass the course.

∴ X will pass the final examination.

(15) If he gambles, then he loses money.

He does not gamble.

∴ He does not lose money.

Use Venn diagrams to determine whether the following syllogisms are valid or invalid and state your conclusion.

(16) Some beatniks are communist.

All beatniks drink coffee.

∴ Some communists drink coffee.

(17) All college students study diligently.

All freshmen are college students.

∴ All freshmen study diligently.

(18) Some athletes are football players.

Joe is an athlete.

∴ Joe is a football player.

(19) Some brunettes are coeds.

All brunettes are good looking.

∴ Some coeds are good looking.

(20) No Democrat eats kumquats.

No Democrat is a Republican.

∴ All Republicans eat kumquats.

- RELATIONS, FUNCTIONS AND GRAPHING -

I. MACHINES AND RELATIONS

Have you ever thought about how a 1¢ bubble gum machine works?

1. You put a penny into the machine.
2. The coin triggers the machine.
3. A ball of bubble gum comes out.

The process can be diagrammed as follows:

Input		Machine		Output
Penny	⟶	Bubble Gum Machine	⟶	Ball of Bubble Gum

What are some other machines which work this way? Example: the juke box machine.

Input		Machine		Output
10¢	⟶	juke box	⟶	music

What would happen if a penny were placed in the juke box? What are other coins which will produce music when put into the juke box? Which coins will not produce music?

Give similar descriptions and discussions about the following "machines":

pay telephone

refrigerator freezer

slot machine

washing machine

integer squaring machine

money changer machine

What are all possible inputs for these machines? What are the outputs which result from each input?

Notice that the so-called machines that we have been discussing are really relationships between two sets of things: the set of inputs and the set of outputs. We refer to these relationships simply as relations; the set of inputs is called the domain, and the set of outputs the range of the relation. For example, the "freezer machine" relates water to ice, milk to ice-milk, etc. The "integer squaring machine" relates the integer 2 to 4, -2 to 4, 3 to 9, etc.

Can you think of any relations which are defined among sets of people?

Examples:

Input	Relation	Output
Fred	likes	Janice
John	has the same height as	Henry
George	is age	______

Can you relate the members of your class using the above three relations? Can you think of other relations with which to classify members of your class or students at your school?

Examples:

_______ was born in the state of _______

_______ comes from the same city as _______

_______ is taller than _______

_______ loves _______

_______ has social security number _______

In the following examples describe appropriate sets of inputs (domains) and their outputs (ranges) for the listed relations. In some cases a sample input or output is provided, and the name of the relation is left blank for guessing. If the relation is among people, you might want to use for the domain, sets of people whom you know personally, such as classmates, family members, etc. For some of the other examples, try to figure out the largest possible domains for the relations.

	Domain	Relation	Range
1.	Boston-New York	distance	____________
2.	____________	fruit squeezer	____________
3.	____________	is the senator of state	____________

	Domain	Relation	Range
4.		weighs	
5.	boy		man
6.		is 1/3 of	
7.	criminal		
8.		xerox machine	
9.		when squared yields	
10.	35°		95°
11.	122°	Fahrenheit--Centigrade	
12.		loves	
13.	New York		8 million
14.		love machine	
15.		is a multiple of	
16.		is larger than	
17.	4, 7	the sum is	
18.	2, 3	the sum of the squares is	
19.	extremist		moderate
20.		River is in the state of	
21.		is the blood relative of	
22.	7	2x + 1	
23.		attends the same school as	
24.	101101		45
25.		is in the same classroom as	
26.	Dr. M.L. King Jr.	social environment	
27.		has the same friends as	
28.	Dr. Ralph Bunche	lived	

	Domain	Relation	Range
29.		is older than	
30.		is the son of	
31.		is the state containing the city	
32.	-6	the square root is	
33.		is a divisor of	8
34.		is the state whose governor is	
35.		base 2	10010
36.	{3,4},{2,3,5}	Set Union	
37.		is one less than	
38.		$2x^2$	8
39.	calve		cow
40.		is faster than	
41.		is twice	
42.	125	$\sqrt[3]{\ }$	
43.		is the cousin of	
44.	6	is one half of	
45.	6	is one half of three times	
46.		is the half brother of	
47.	1, -3	the sum of the cubes is	
48.	{3,4},{2,3,5}	set intersection	
49.	12	squared minus 3 yields	
50.		squared minus 7 yields	

Exercises

A. General Questions On The Examples

1. If the members of your class are represented by dots such as below, can you figure out a way of diagramming the relations: "live in the same city", "is taller than", "likes", "have the same major"?

2. How would you diagram or graph other relations defined among members of your class?

3. Diagram some of the family relations such as "son of", "brother of", etc., defined on members of your family.

4. How could you diagram some of the relations whose domains are different from their ranges?

5. Examine the relations in 1 - 50 above and pick out those that are similar to each other. How are they similar? Can your diagrams help?

B. More Specific Questions On The Examples

1. Which of the relations have their domains and ranges identical?

2. For which of the relations is there some confusion about which input is related to which output?

3. For which of the relations is it true that some inputs are related to more than one output?

4. Which of the relations have more than one input associated with a single output?

5. For which of the relations is it true that each input is associated with exactly one output?

II. INVESTIGATION ON EQUIVALENCE RELATIONS

Materials: A set of attribute blocks and some pattern blocks.
Procedure: Select a subset of these blocks ---

Are all of these blocks related? Can you separate these blocks into subsets, such that, all blocks in the same subset are related?

How are these blocks related?

For example: (1) "has the same color as," (2) "has the same shape as," (3) "one side the same length as," (4) "one angle the same size as," (5) "has a larger surface area than," (6) "not the same color as," etc.

What are some other properties that can be used to relate the blocks? (For each of the relations, try to arrange the blocks so that the blocks which are related are grouped together.)

1. Do all of the relations divide the blocks into disjoint subsets (the blocks in each subset are all related, but are not related to the blocks in any other subset)?

2. Classify the relations according to whether they divide the blocks into disjoint subsets or not.

3. What are some properties that are true of those relations that divide (do not divide) the blocks into disjoint subsets?

   (a) Is each block related to itself?

   (b) If a first block is related to a second block by this relation, is the second block also related to the first block by the same relation?

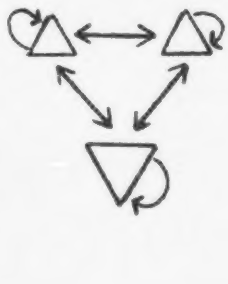
   (c) If a first block is related to a second block and a second block is related to a third block by the same relation, is the first block related to the third block by that relation?

4. Classify the relations that do not divide the blocks into disjoint subsets according to the properties that they satisfy.

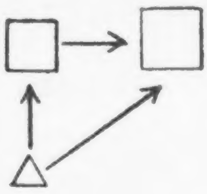
Diagramming Relations

Can you find a way of representing the above relations? How would you indicate that a block is related to itself? How would you indicate that two blocks are related to each other? (Ans: One way is to draw geometric figures to represent the blocks and then draw arrows between the figures which represent blocks that are related.) What are the relations in the following examples?

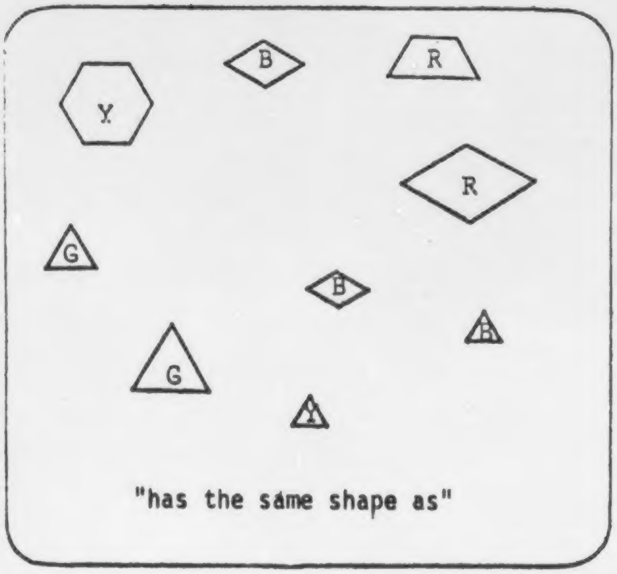
(i)



(ii)

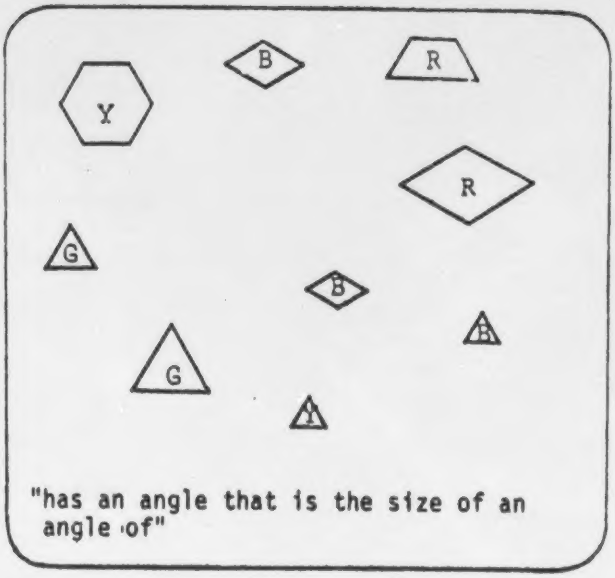


Indicate with arrows the shapes that are related as specified in the following:



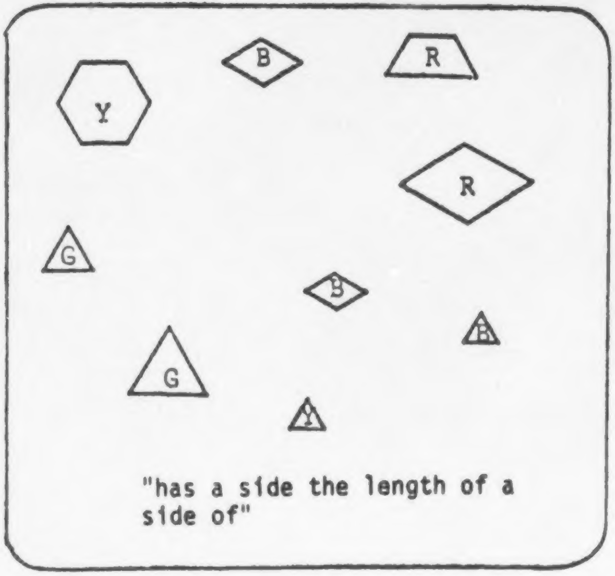
"has the same shape as"

Diagram 1



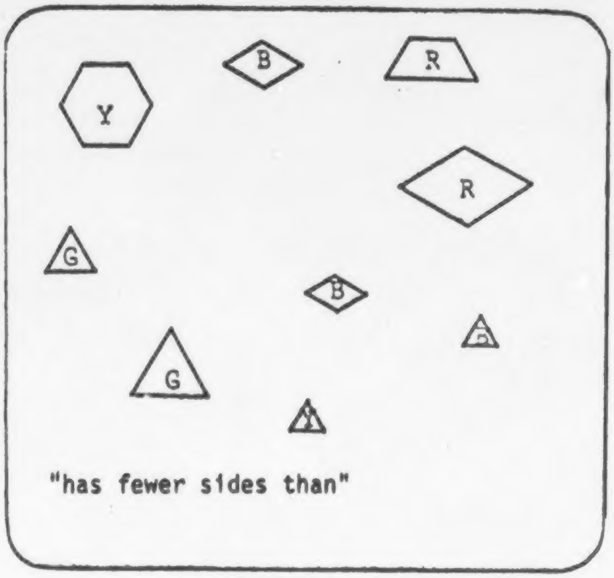
"has an angle that is the size of an angle of"

Diagram 2



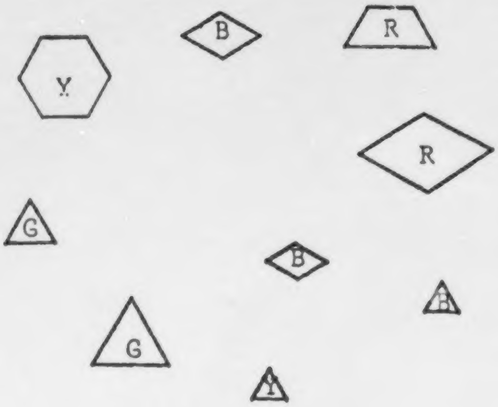
"has a side the length of a side of"

Diagram 3



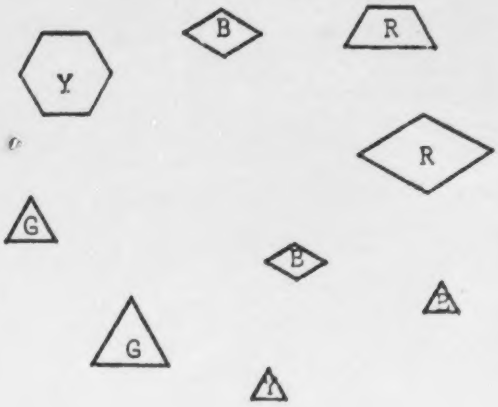
"has fewer sides than"

Diagram 4



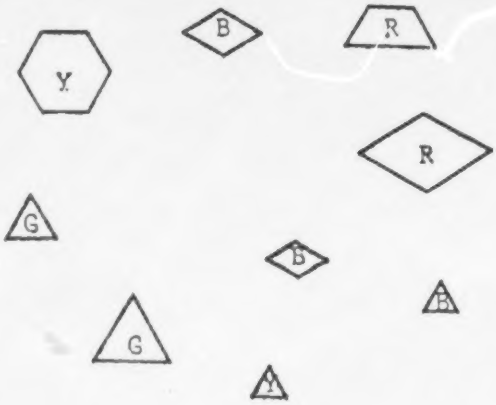
"has fewer angles than or the same number of angles as"

Diagram 5



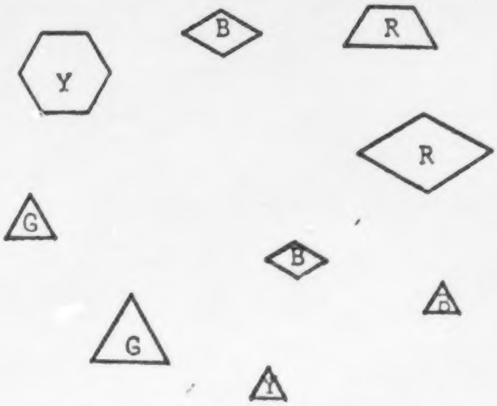
"has the same perimeter as"

Diagram 6



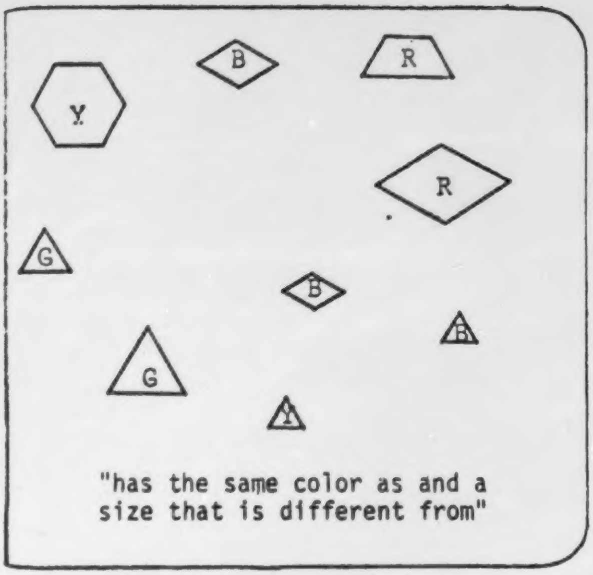
"has no angle the same size as an angle of"

Diagram 7



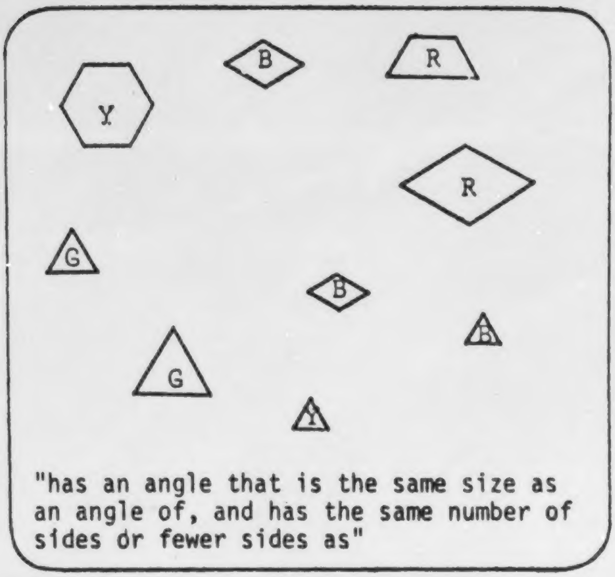
"has the same shape as and a color different from"

Diagram 8



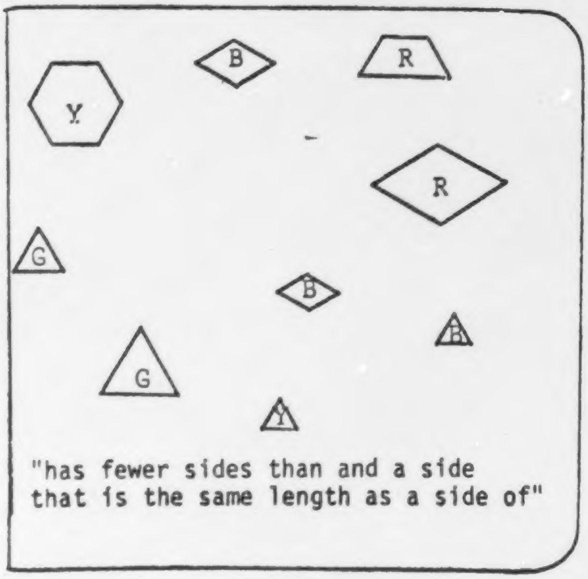
"has the same color as and a size that is different from"

Diagram 9



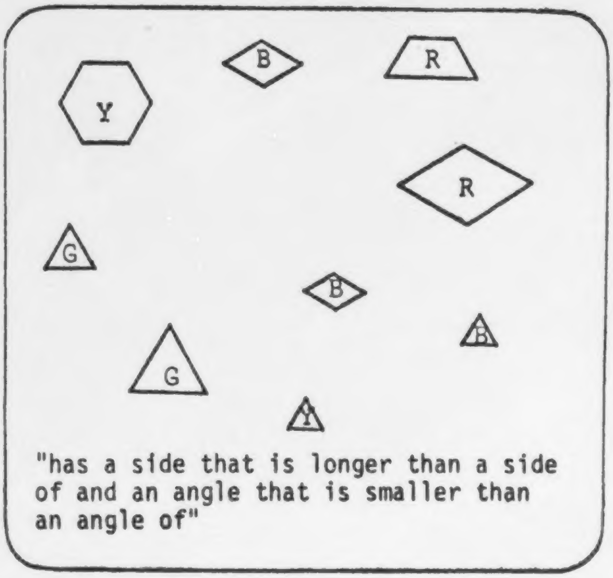
"has an angle that is the same size as an angle of, and has the same number of sides or fewer sides as"

Diagram 10



"has fewer sides than and a side that is the same length as a side of"

Diagram 11



"has a side that is longer than a side of and an angle that is smaller than an angle of"

Diagram 12

Test each of the previous diagrams to see if it satisfies any of the following three properties:

(R) Each figure has an arrow connecting it with itself.

(S) Any two figures which are connected by an arrow in one direction, are connected by an arrow in the other direction also.

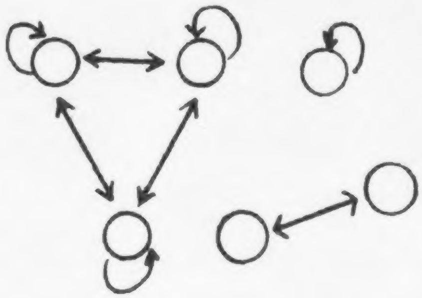
(T) If there is an arrow pointing from one figure to a second, and an arrow pointing from the second figure to a third figure, then there is an arrow pointing from the first figure to the third figure.

- Which of the diagrams satisfy R? which satisfy S? which satisfy T?
- Which satisfy R only? S only? T only?
- Do any satisfy R and S, but not T? R and T, but not S? etc.
- If a diagram satisfies all three of the properties, what is true of the figures in that diagram?
- What can you say about a diagram which has the figures divided into disjoint subsets? Which of R, S, and T do these particular diagrams satisfy?

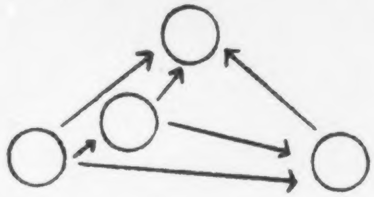
General Observations

Suppose you are considering relations on the students in your classroom. One way of diagramming such relations is to represent the students with dots or little circles and to use arrows as in the preceding diagrams.

(i)



(ii)



Can you guess what relations diagrams (i) and (ii) might represent? Some possibilities are "come from the same city" and "is taller than". What are some other possibilities? Examine several such relations and test to see which of them have diagrams satisfying R, S and T.

Based on the examples you have considered, can you guess what properties any relation defined on any set must have it its diagram is to satisfy R, S and T?

You probably guessed the following properties:

(R) Each element is related to itself.

(S) If an element is related to a second element, then the second is also related to the first.

(T) If an element is related to a second element, and the second is related to a third, then the first is also related to the third.

Any relation which satisfies R, S and T is called an equivalence relation. The letter R stands for reflexive; S stands for symmetric; T stands for transitive. Can you guess why this choice of words is used?

You have probably discovered from the examples that an equivalence relation on a set divides the set into disjoint subsets so that the elements in each subset are inter-related and are not related to elements in any other of the subsets. These disjoint subsets are called equivalence classes.

- Which of the diagrams that you constructed are the diagrams of equivalence relations? Pick out the equivalence classes in these diagrams.
- Which of the relations in 1 - 50 in Part I are equivalence relations? Describe the equivalence classes in each case.
- What must be true about the domain and the range of a relation which is an equivalence relation?

Exercises

1. Investigate the following relations defined on the attribute blocks:

______ has a larger surface area than ______

______ has a surface at least as large as ______

______ has a different shape than ______

______ has a different shape but the same color as ______

______ has larger surface area and same color as ______

______ is touching ______

______ is pointing at ______

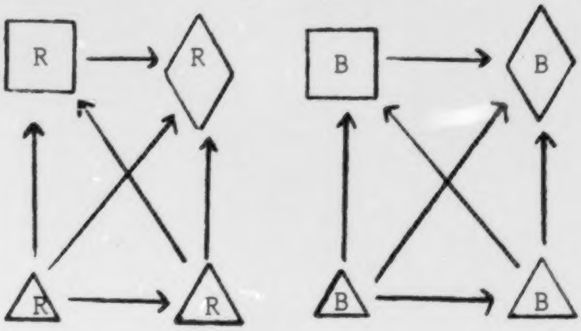
____and____have a total number of sides which is an even number

____and____have a total number of sides which is an odd number

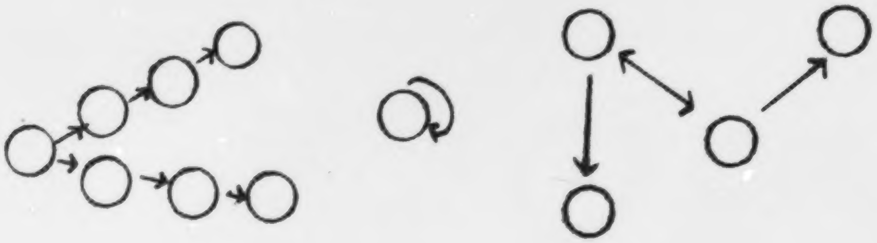
2. Which of the relations in 1 are equivalence relations?

3. Sketch a diagram which is transitive only. How would you have to alter your diagram to make it also symmetric?

4. Can you guess which relation is depicted in the follow diagram?



5. Which of the properties R, S, T does the following diagram satisfy?



How must you alter this diagram to make it the diagram of an equivalence relation?

6. Which of the family relations, "son of", "brother of", "blood relative of", "father of", etc., are equivalence relations?

7. Is the relation, "have the same boy friend", among girls in your classroom (or at your school) an equivalence relation? How about "have the same girl friend" among the fellows?

8. Are the relations:

   congruent to
   similar to
   parallel to
   perpendicular to
   supplement of
   equal to

   equivalence relations?

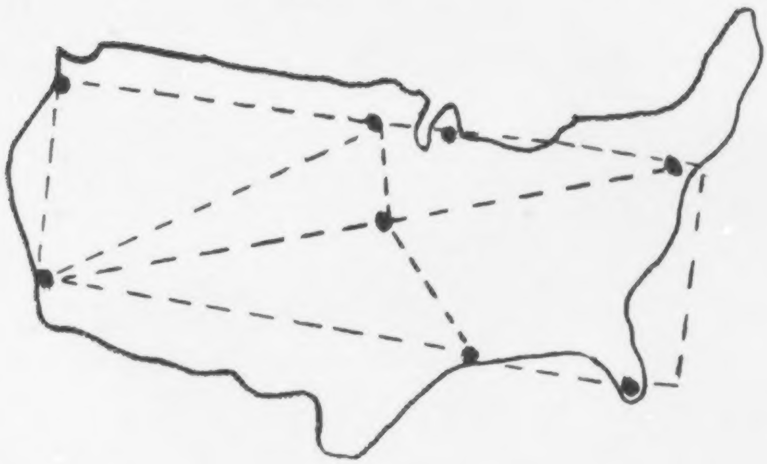
9. Consider the following relationship among ten follows:

   Frank knows Joe
   Bob knows Andy
   Tom knows Larry
   John knows Hank
   Ted knows Frank
   Andy knows John
   Hal knows Larry

   (Assume that each fellow makes certain that all of his friends know one another.)

   Is this an equivalence relation? Can you break the follows up into groups so that those in the same group know each other but do not know those in any of the other groups? If so, how many groups do you get in this way? Are these "groups" equivalence classes?

10. Below is a map of the United States showing the air routes maintained between major cities by one of the airline companies:



This airline company wants to man its airport control towers with eight men, one at each tower. It would be helpful if the company put men who knew each other in cities with air routes between them. Below are the pairs of men who are friends:

Frank knows Tim
Ken knows Don
George knows Al
Don knows Al
Tim knows Larry
Frank knows George
Larry knows Don
Ken knows George
Frank knows Ted
Ted knows Tim
Frank knows Ken
Tim knows Ken

Can you show how the company might station their eight men in the eight airport towers to satisfy this condition? (It _is_ possible.) Is there more than one way to do this? If not, _why_ not?

Is the relation you have set up an equivalence relation? If not, can you alter it in any way so that it becomes an equivalence relation?

11. In a town there are seven men some of whom know each other, some of whom do not. Below are listed pairs of men who are friends?

Fred knows Sam
John knows Bob
John knows Bill
Fred knows Bob
Fred knows John
Sam knows Bob
Tom knows John
Sam knows Tom
Sam knows John
Sam knows Bill
John knows Andy
Tom knows Bob

(Whenever "Fred knows Sam", it is assumed that "Sam knows Fred" too.)

In the same town there are seven women. Below are listed pairs of women who know each other:

Mary knows Joan
Bea knows Ruth
Carol knows Ruth
Betty knows Bea
Mary knows Betty
Carol knows Mary
Ann knows Mary
Ann knows Ruth
Betty knows Ann
Ruth knows Mary
Mary knows Bea
Ruth knows Betty

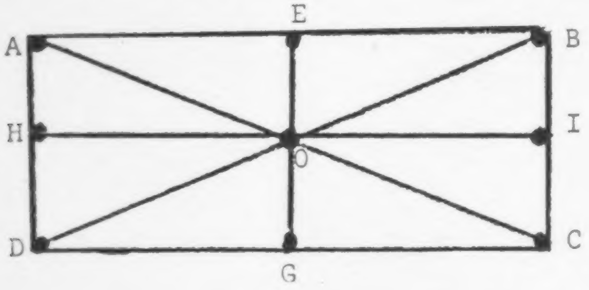
The question is: Can you marry off the men to the women in such a way that whenever two men know each other, their wives will too, and vice versa; and so that whenever two men do not know each other, their wives don't either?

Are there any diagrams you can draw that might help you to solve this problem?

Is your relation an equivalence relation?

III. EQUIVALENCE RELATIONS EXERCISES

1. The questions in this exercise refer to the following figure:



Assume that it is allowable to trace a path from any one of the labeled points to another along any of the lines in the diagram.

A. For any two of the labeled points X and Y we define the relation "Point X is related to point Y" if a path can be traced from X to Y which passes through exactly one labeled point besides X before arriving at Y. (Retracing is allowed.)

   1. What is the domain and range of this relation? Can you diagram it?

   2. Is the relation reflexive? symmetric? transitive? Is it an equivalence relation? If so, describe the equivalence classes.

   3. Answer all the above questions if retracing is not allowed.

B. Now, suppose we alter the relation in A by requiring that a path from X to Y must pass through exactly two intermediate points with retracing allowed. Answer questions 1 through 3 above for this relation.

C. Suppose that a path can be traced between two points only if the points lie on the same segment and have exactly one intermediate point between them. Answer questions 1 through 3 above for this relation.

2. Diagram the following relations defined on the set of positive integers and indicate which of them are equivalence relations. Also, describe the equivalence classes:

   less than

   less than or equal to

   divisor of

   multiple of

   _____ and _____ are related if their difference is divisible by 3

   _____ and _____ are related if they differ by 3

   _____ and _____ are related if the ratio of the first to the second is a whole number

   _____ and _____ are related if their sum is odd

   _____ and _____ are related if their sum is even.

3. Can you find any relations which are reflective and transitive but not symmetric? symmetric and transitive but not reflexive? reflexive and symmetric but not transitive?

   How do the diagrams of such relations differ?

4. Which of the relations 1 - 50 in Part I are neither reflexive, symmetric nor transitive? What, besides not satisfying the three properties, do these relations have in common? How would you diagram them?

IV. FUNCTIONS

Among the relations 1 - 50 in Part I, pick out those that satisfy the following conditions:

Each input in the domain is related to exactly one output in the range.

Relations which satisfy this condition are usually referred to as functions. Some examples are:

the social security relation

was born in the state of

the integer squaring relation

What is the largest domain that you can think of for each of these three examples? Describe the associated ranges.

A handy way of diagramming functions is to construct tables such as:

- social security -

Domain	James	Harold	...
Range	283-34-3517	295-38-4529	...

- was born in the state of -

Domain	Sally	Joe	Frank	. . .
Range	Texas	Alabama	Ohio	. . .

- squaring -

Domain	1	2	3	4	. . .
Range	2	4	6	8	. . .

Can you think of other ways of diagramming a function? How about using circles and arrows as with some of the relations in Part II?

Consider the relation which associates with each student in your class his or her top three favorite soul artists. Is this relation a function? How would you diagram it? How does the diagram of a function differ from the diagram of other relations?

In the examples below try to guess which functions are tabulated.

(1)

Domain	Joe	Harry	Tom	Jake	Sam
Range	21	16	27	32	18

(2)

Domain	calve	colt	puppy	cub		cow	. . .
Range	cow	horse	dog		deer		

(3)

Domain	1	2	3	4	. . .
Range	2	3	4	5	. . .

(4)

Domain	1	2	3	4	5	6	. . .	51	. . .
Range	2	3	5	5	7	7		53	. . .

(5)

Domain	1	2	3	4	5	. . .	29	. . .
Range	1	3	5	7	9			

(6)

Domain	-4	-2	0	4	5	. . .
Range	-2	-1	0	2	$\frac{5}{2}$	. . .

(7)

Domain	1	2	3	4	5	6	7	. . .
Range	0	1	0	1	0			

(8)

Domain	-3	-2	-1	0	1	2	3						
Range								-3	-2	-1	0	1	2

(9)

Domain	1	2	3	4	5	. . .
Range	1	3	6	10	15	. . .

Illustration

In number 3 above, you probably guessed that the function associates with each number the number that is one larger; so that with 2 it associates 2 + 1; with 3, it associates 3 + 1, etc. Thus, if x represents any of the inputs in the domain, the associated output is x + 1. In many books this particular function is written as an equation, $y = x + 1$, where x stands for input and y stands for the associated output. Can you write equations for any other functions in the previous examples?

Describe the functions which are given by the following equations. Indicate appropriate sets for the domain.

$y = 3x + 1$ $\qquad$ $y = x^2 + 1$

$y = x - 7$ $\qquad$ $y = x^2 - 1$

$y = x$ $\qquad$ $y = \sqrt{x}$

Do you recall how to draw pictures (make graphs) of equations like these using graph paper? Sketch graphs or pictures of all the above functions using graph paper. (It might be necessary for you to review graphing at this point. You could consult elementary mathematics textbooks or see the TCCP unit "Graphing".)

Exercises

1. Boys: Joe, Hal, Tom, Bob, Fred, Lou, Richard, Jim

   Girls: Mildred, Gladys, Mary, Janie, Lucy, Sarah

   Can you set up a relation between the girls and boys so that each boy has 3 girl friends and each girl has at least one boy friend?

   Can you alter your relation so that each girl marries exactly one of her boy friends?

   Which, if any, of the relations you have constructed are functions?

2. Write equations for the functions whose tables are given as follows:

Domain	-4	-3	-2	-1	0	1	2	. . .
Range	16	9	4	1	0	1	4	. . .

Domain	0	1	2	3	4	5	. . .
Range	1	4	7	10	13	16	. . .

3. Consider the relation which associates with each fraction the smallest whole number which is larger than the fraction. Is this relation a function? If so, can you make a table describing it? What does its graph look like?

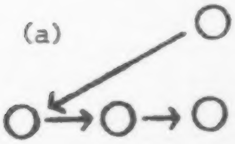
4. Graph the relation which associates with each positive integer the number of divisors it has. Is this graph a straight line? Why?

5. Graph the relation which associates with each positive integer its divisors. Is this a function?

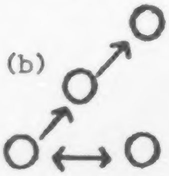
6. Can you tell from its graph whether a relation is a function?

7. Which of the following are diagrams of functions?

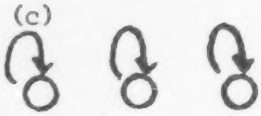
(a)



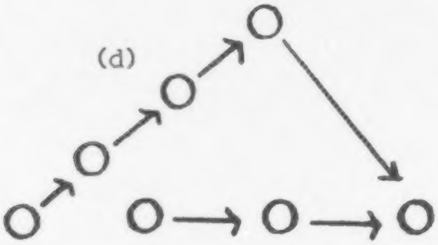
(b)



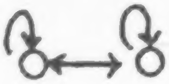
(c)



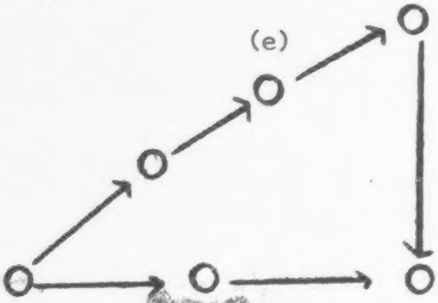
(d)



(f)



(e)



For those which are not functions, can you alter the diagrams in any way to make them functions?

8. Are any of the diagrams in 5 the diagrams of equivalence relations?

9. Can you find any equivalence relations which are also functions? Any equivalence relations which are not functions? Any functions which are not equivalence relations?

10. Which of the relations in exercises 9 and 10 of Part II are functions?

V. FUNCTIONS EXERCISE

Suppose there is a grasshopper sitting on a number line and it jumps according to the following pattern:

Initial Position	Final Position
0	5
1	7
2	9
3	11
4	13

(a) Where would the grasshopper land if it starts from 5? 6? 7? -1? -5? -15?

(b) From which point should it start if it wants to land at 0? 75? -11? 6?

(c) Where would the grasshopper land if it starts at n?

(d) There is another grasshopper which first jumps according to the rule $n \rightarrow (2n-5)$, and then it jumps according to the rule $n \rightarrow (2n+5)$. If it starts at 15, where will it land after the two jumps.

Fill in the diagram below.

$$\boxed{15} \xrightarrow{2n-5} \boxed{} \xrightarrow{2n+5} \boxed{}$$

(e) Do the same for the following diagrams:

$$\boxed{20} \xrightarrow{2n-5} \boxed{} \xrightarrow{2n+5} \boxed{}$$

$$\boxed{25} \xrightarrow{2n-5} \boxed{} \xrightarrow{2n+5} \boxed{}$$

$$\boxed{30} \xrightarrow{2n-5} \boxed{} \xrightarrow{2n+5} \boxed{}$$

(f) Now make the table for the results obtained in (e) and (f).

Initial Position	Final Position
15	☐
20	☐
25	☐
30	☐

Can you make a rule which will take the grasshopper from initial position into final position?

Exercises

1. In the following tables x denotes elements of the domain and y denotes elements of the range of some functions.

x	y
-2	-6
-3	-3
0	0
1	3
2	6
3	9
4	12
5	15

x	y
-2	-4
-1	-1
0	2
1	5
2	8
3	11
4	14
5	17

x	y
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16
5	25

x	y
-2	3
-1	0
0	-1
1	0
2	3
3	8
4	15
5	24

x	y
-2	-8
-1	-1
0	0
1	1
2	8
3	27
4	64
5	125

x	y
-2	-7
-1	0
0	1
1	2
2	9
3	28
4	65
5	126

Try to find equations in x and y which describe these functions and graph the functions. What can you conclude from the graphs?

2. Let $X=\{1,2,3\}$. Find all possible functions from x to x.

3. A set of ordered pairs such as $A=\{(2,1), (3,1), (4,2), (6,7)\}$ defines a relation. We say that "a is related to b" if the pair (a,b) is in A.

   For each of the following relations tell whether or not it is a function.

   $A=\{(2,1), (3,1), (4,2), (6,7)\}$

   $B=\{(3,3), (4,7), (8,9), (3,5), (1,2)\}$

   $C=\{(3,3), (4,5), (6,9), (7,1), (5,4)\}$

   $D=\{(1,1), (2,1), (1,3), (1,4), (1,5)\}$

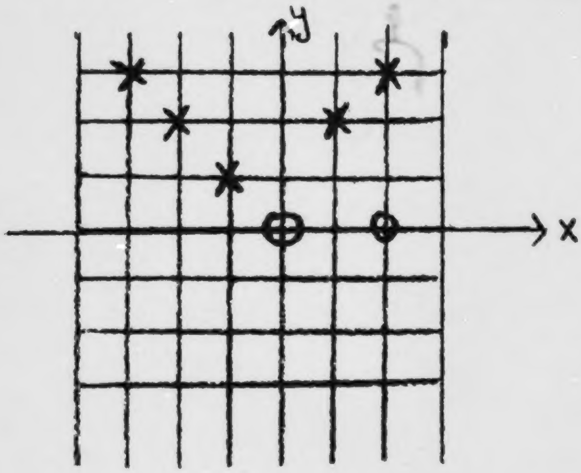
   Graph each one.

4. An integer y is related to an integer x so that y is 3 less than twice x. State the relation as a set of ordered pairs for $-3 < x < +3$.

5. The domain of a set of ordered pairs is the set $\{1, 2, 3, 4, 5\}$ and the equation defining the relation is $y = 3x - 2$. Write the set of ordered pairs.

VI. TIC-TAC-TOE INVESTIGATION

A variation of the game Tic-Tac-Toe can be played on graph paper by making X's and O's on the graph paper as indicated below.



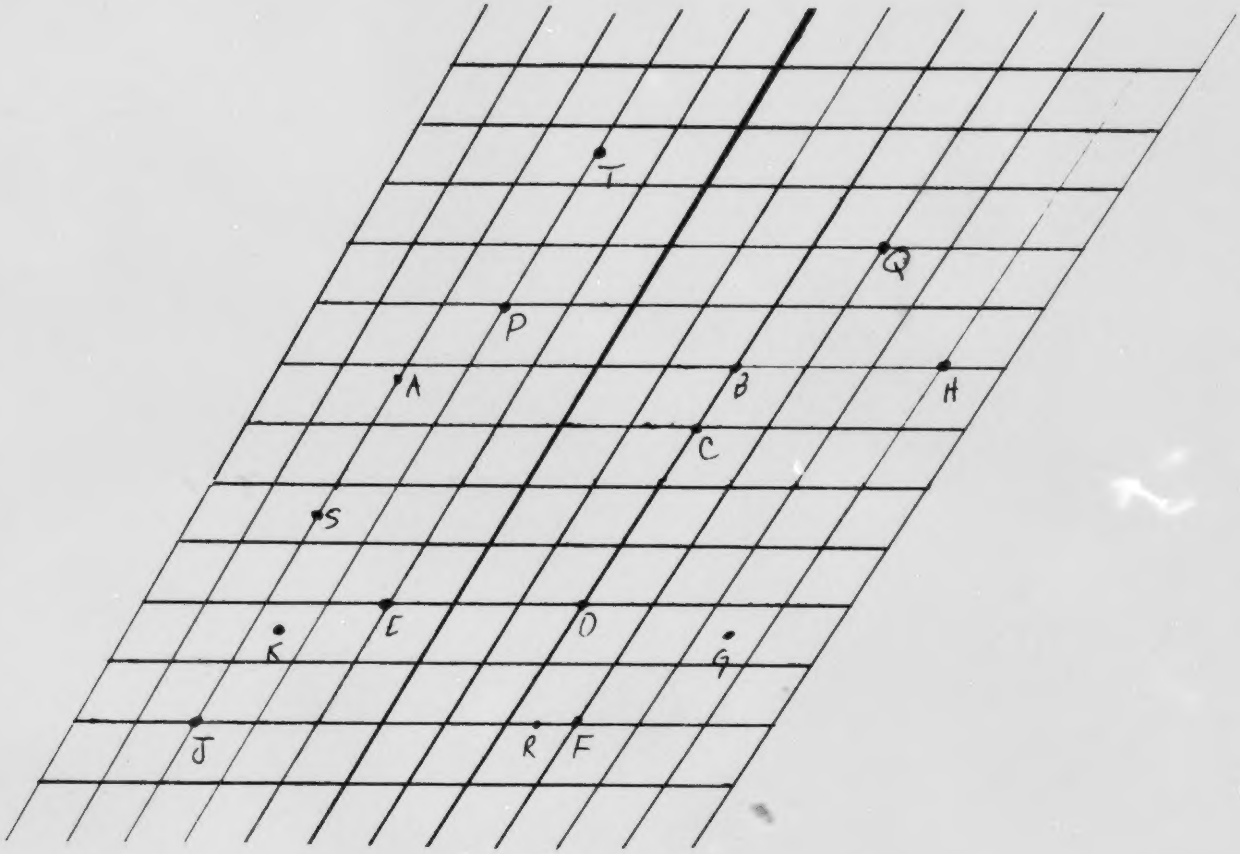
Play several such games with a classmate using the rules: four X's in a row are needed to win; X's and O's are marked only on grid points.

The rules and tactics for this game are flexible as indicated by the following possibilities:

> Although you may wish to begin in the first quadrant, the other quadrants bring up questions about negatives that can be easily dealt with in the context of this game.
>
> Requiring each player to name the position of his move focuses initial attention upon coordinates. When the players become fairly proficient, add the rule that the first position named counts.
>
> Later you may want to require the winner to give the correct equation for his winning "line" in order to be officially declared the "winner".
>
> Restricting the size of the grid to 5 x 5, will require more strategy. A well played game on a 5 x 5 grid will always lead to a draw. If the grid is made 6 x 6, four in a row to win, there is a winning strategy for the player taking the first move.

Exercises

1. Suppose you know that in the following coordinate system the point C has coordinates (2,1).

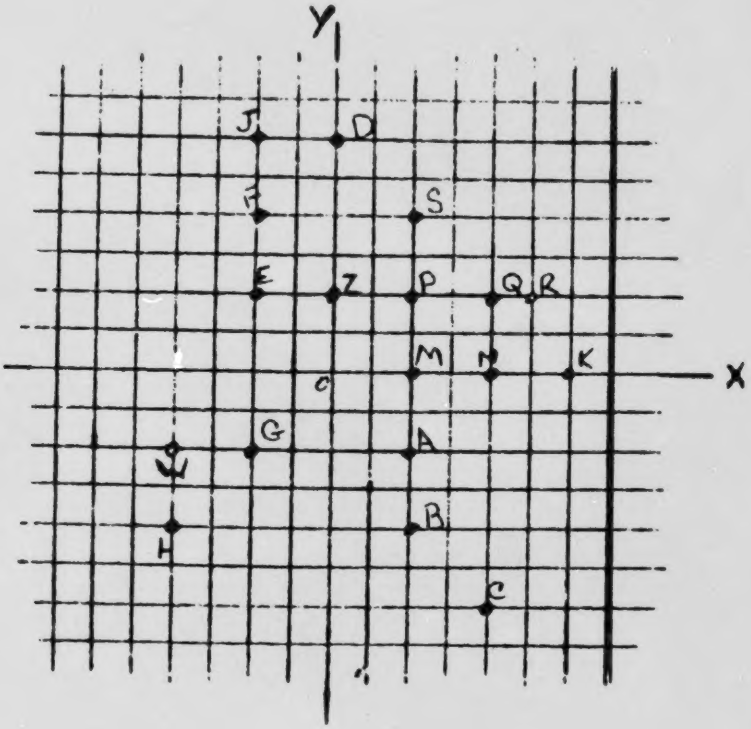


Locate the coordinate axes.

Find coordinates for the other points:

A.
B.
C.(2,1)
D.
E.
F.
G.
H.
J.
K.
P.
Q.
R.
S.
T.

2. Suppose you know that in the following coordinates system the point E has coordinates (-4,2).


Y
J
D
F
S
E
Z
P
Q
R
M
N
K
X
O
G
A
W
H
B
C

Can you find coordinates for the other points?

After you have played several games of Coordinate TIC-TAC-TOE with your classmate, try to answer the following questions which refer to that game.

3. Consider the following combinations. Which are winning combinations?

x	y
-1	3
0	5
1	7
2	9
3	11

x	y
-3	1
-2	2
-1	4
0	8
2	16

x	y
-1	1
-2	2
-3	3
-4	4
-5	5

Answer the following questions for each table:

If x = 4, what is y?

If x = 5, y = _____.

If x = 10, y = _____.

Can y = 8?

What if y = 19, what is x?

If y = 1, x = _____.

If y = -1, x = _____.

If y = -3, x = _____.

4. Take turns with your partner naming values of x and see if he can correctly name y. Try it with naming y first. What rules did you discover?

5. Now consider the following combinations. Which ones have patterns? _________. Which ones are winning combinations? _________. Do all the winning combinations have patterns? ____yes ____no. Are all the ones with patterns winning combinations? ____yes ____no.

(1)

x	y
-1	1
0	2
1	3
2	4
3	5

(2)

x	y
-3	3
-2	4
-1	5
0	6
1	7

(3)

x	y
-2	-3
-1	-2
0	-1
1	0
2	1

(4)

x	y
1	0
2	1
3	2
4	3
5	4

(5)

x	y
1	0
2	1
3	2
4	3
5	4

(6)

x	y
4	8
3	6
2	4
1	2
0	0

(7)

x	y
2	4
3	5
4	6
5	7
6	8

(8)

x	y
0	0
1	3
2	6
3	9
4	12

(9)

x	y
-3	3
-2	2
-1	1
0	0
1	-1

(10)

x	y
-2	6
-1	9
0	6
1	9
2	6

(11)

x	y
0	5
1	4
2	3
3	2
4	1

(12)

x	y
-3	-1
-2	1
-1	3
0	5
1	7

(13)

x	y
-1	1
-8	4
-7	7
-6	10
-5	13

(14)

x	y
3	2
4	1
5	-1
6	3
7	-4

(15)

x	y
3	8
4	7
5	6
6	5
7	4

(16)

x	y
2	1
3	2
4	4
5	7
6	11

(17)

x	y
1	2
2	4
3	3
4	-10
5	7

(18)

x	y
-3	8
-2	7
-1	6
0	5
1	4

(19)

x	y
4	10
3	7
2	4
1	1
0	-2

(20)

x	y
1	6
2	6
3	6
4	6
5	6

Find some pairs of combinations with the same pattern. Could you ever tell that a combination was a winning one from its pattern before you drew it?

6. With your partner, make up some of your own rules of correspondence so that when he gives you a number you will use that number as x and tell him y. See how soon he can figure out your rule.

   Now write down some of your rules and make a table of some of the pairs. Then plot the points in your table. Which ones look like they will be straight lines?

7. Write down the rules of correspondence for each of the 20 tables which have patterns.

   Make a table and draw the graph of each of the following:

(a) $y = x + 3$

(b) $y = 2x$

(c) $y = 2x + 3$

(d) $y = 6$

(e) $y = 4 - x$

(f) $x = 3$

8. (a) Plot the points $P_1(2,3)$ and $P_2(4,7)$ on graph paper. Measure how far in the horizontal direction, H, and how far in the vertical direction, V, you have to go to get from P_1 to P_2. Notice in this case that

$$V/H = 4/2 = 2$$

(b) Find a third point where horizontal distance H from P_1 and where vertical distance V from P_1 are such that $V/H = 2$.

How many such points can you find? Graph and describe the set of all such points. What geometric figure do you get? (Straight line)

(c) If you could extend your graph paper indefinitely, would the point (115,200) be on your line? Can you write an equation which a point (x,y) must satisfy in order to be on the line? What is the equation of the line?

(d) Repeat the above problems starting with other pairs of points, such as $P_1(-1,3)$, $P_2(2,-1)$, etc. What conclusions can you draw? What can you say in general about the equations of straight lines?

9. (a) Graph the equation $y = 2x + 1$. Select two points on this graph and calculate V/H as in previous exercises. Do the same for the equations:

$y = 2x$ $\qquad$ $y = 2x - 1$

$y = 2x + 4$ $\qquad$ $y = 2x - 4$

$y = 2x + 6$

For a given line the fraction V/H, as defined in A, is referred to as the slope of the line. What can you say about the slopes of the above lines?

(b) Repeat exercise 1 for the lines:

$y = -3x + 1$ $\qquad$ $y = -3x - 4$

$y = -3x + 7$ $\qquad$ $y = -3x$

$y = -3x - 2$

(c) If the slope of a line is positive, which way does the line slant?

If the slope of a line is negative, which way does it slant?

If you want a line to be steeper, how must you change the slope?

What do you notice about the slopes of vertical and horizontal lines?

(d) Find the slopes of the following lines:

(a) $y = x + 5$

(b) $y = -2 + 7$

(c) $y + 3 = x$

(d) $x - y = 5$

(e) $4x + 2y = 1$

(f) $y = 4 - \frac{x}{2}$

10. Look at your graphs in B and determine where each of the lines crosses the x and y axes. Can you tell by looking at the equation of a line where it will cross the x and y axes?

VII. MIRROR GRAPHS

Draw a straight line segment passing through the points (0,0), (1,1), (2,2), (3,3).

a) Place a mirror on the X axis and draw the mirror image of the line.

Write the coordinates of several points on the mirror image line. Find a relationship between the new points and the old points.

b) Put the mirror on the Y axis and draw the mirror image of the line.

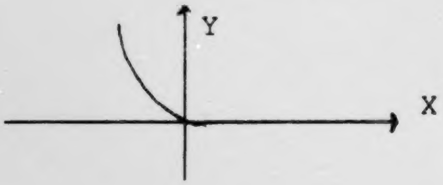
Write the coordinates of several points on the mirror image line, and find a relationship between the new points and the old points.

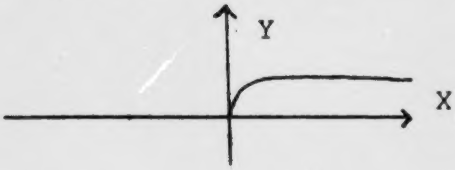
c) Where would you place the mirror so that the mirror image of the segment appears to be in the third quadrant? How do the coordinates of points on this mirror image differ from the old coordinates?

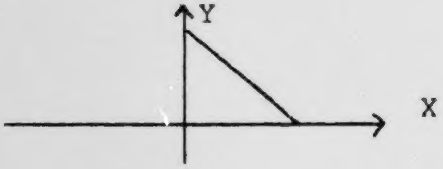
d) Where would you place the mirror so that the mirror image of the line coincides with the line itself?

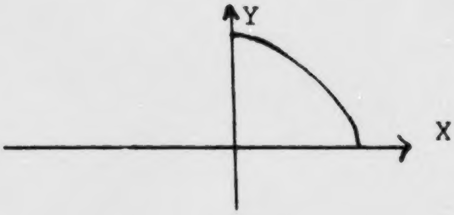
Exercises

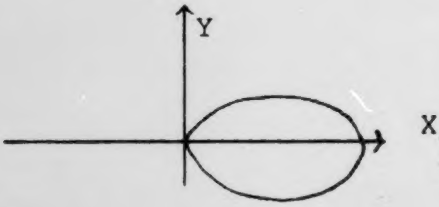
1. Describe the mirror images of the following figures if the mirror is placed on the X axis; on the Y axis; at an angle of 45° with the X axis. (First make guesses without actually using the mirror, then test your guesses with the aid of the mirror.) In each case, try to write a mathematical description of the mirror images.

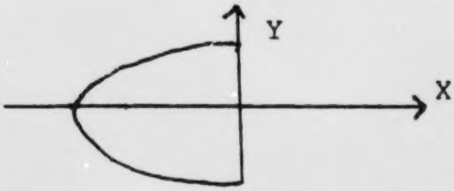




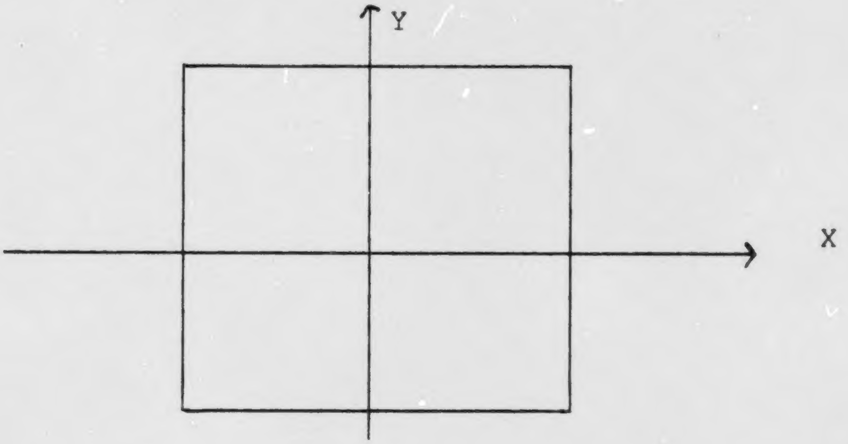








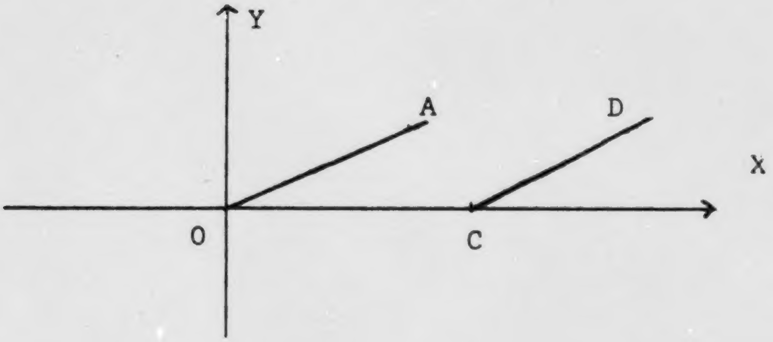
2. In the figure below, put the mirror at angles 30°, 45°, 60°, 75° with the X axis. Describe the resulting images.



a) In each of the images, describe what happens to perpendicular and parallel lines.

b) Where would you put your mirror so that the square will look bigger (smaller) then what it is?

3. In the figure below, where would you put a mirror so that the line OA changes to line CD?



VIII. LOONEY GRAPHING

A. The following set of ordered pairs forms a name or makes a pattern. Plot the points and try to guess the name or pattern. (Hint: a real woman killer.)

1. (0,0)
2. (5,2)
3. (-4,3)
4. (-2,0)
5. (2,3)
6. (-2,3)
7. (6,3)
8. (4,0)
9. (4,3)
10. (0,3)
11. (2,0)
12. (-4,0)

It might help if you connect some of the points with straight lines.

B. Try to guess the name or pattern found by the pairs:

1. (-4,0)
2. (-2, 1 1/2)
3. (4,0)
4. (2,3)
5. (0,0)
6. (0,3)
7. (-3, 1 1/2)
8. (4,3)
9. (2,0)
10. (1,3)
11. (-1,3)
12. (-2,3)
13. (-4,3)
14. (-2,0)

C. Do the same for:

1. (3,0)
2. (0,3)
3. (4.5,5)
4. (1,7)
5. (6,7)
6. (3,7)
7. (2,8)
8. (7,8)
9. (8.7)
10. (7,11)
11. (9,13)
12. (9,3)
13. (0,13)
14. (2,11)
15. (4,4)
16. (3,3)
17. (5,4)
18. (6,3)
19. (6,0)

D. Write down the coordinates which will form other names or patterns and try them out on your classmates.

Exercises

1. Graph the set of all points in the plane whose distance from the point P(0,0) is exactly two units. What kind of figure have you graphed? If (x,y) is any point on the figure, can you write an equation which expresses the fact that (x,y) is two units from (0,0)? Does this equation describe the graph you made?

2. Graph the set of all points in the plane whose distance from P(2,1) is exactly two units. What is the equation of this graph? How does it compare with the equation in 1?

3. Try several other problems similar to 1 and 2 by changing the point P and the given distance from P. Any conclusions?

4. Graph the set of all points in the plane whose distance from P(0,4) is the same as its perpendicular distance from the x axis. What is the equation of this graph? If y is replaced by negative y in the equation, what happens to the graph? What are some other changes that can be made in the equation which will alter the position of the figure but not the shape?

Compare the graphs of the equations:

i) $x^2 + y^2 = 4$

ii) $(x-2)^2 + y^2 = 4$

iii) $(x-2)^2 + (y-2)^2 = 4$

Answer the following questions about each of the graphs in 1.

(a) What is the largest (smallest) value for x?

(b) What is the largest (smallest) value for y?

(c) What geometric figure is determined by the graph?

(d) Write the equations of the lines which determine the smallest rectangular bond for the figure.

Describe the graphs of the following and try to draw some general conclusions about graphs and equations of circles.

(a) $x^2 + y^2 = 25$

(b) $2x^2 + 3y^2 = 9$

(c) $x^2 + (y-5)^2 = 16$

(d) $(x+3)^2 + y^2 = 36$

(e) $(x-3)^2 + (y+2)^2 = 9$

(f) $x^2 + y^2 = -4$

(g) $(x+h)^2 + (y+k)^2 = 0$

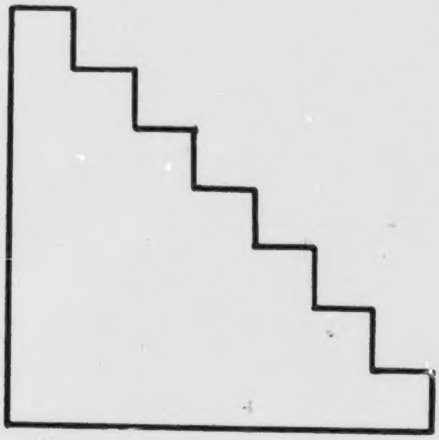
(h) $3(x-2)^2 + 5(y+3)^2 = 15$

- TOOLS AND CONCEPTS -

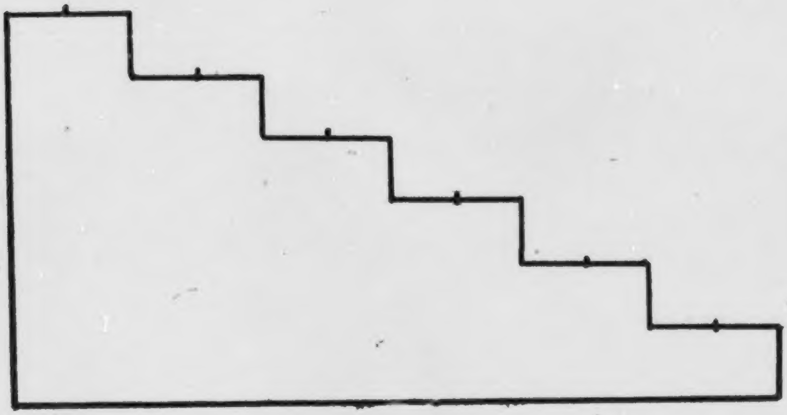
I. SEQUENCES

I. If a block is of this size how many blocks will it take to make the following staircases?

(a)



(b)



Suppose you want to make these staircases one step higher, how many blocks will you need?

Try some of your own staircases.

Build the simplest staircase of type (a) you can think of. Now, keep building it up one step at a time and fill in the following table: (the starred boxes will be easier after the next exercise)

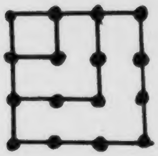
Number of steps	Number of blocks added: type (a)	Total number of blocks needed: type (a)	Number of blocks added: type (b)	Total number of blocks needed: type (b)
1				
2				
3				
4				
5				
10				
15		*		*
143		*		*
179		*		*
4963		*		*

After doing the following with your partner, you should be able to fill in the starred (*) boxes. Each of you first needs to construct the staircase(b). Now, fit the two staircases together to form a rectangle. Do you see an easy way to find the total number of blocks in the two together? What is it? What must you do to find the total used in one staircase?

Do this for other types of staircases. Does it always work? Describe the type of staircases for which this method workds. Now fill in the remaining boxes in the table.

Exercises

1. Set up a sequence of squares that share one corner of the geo board. Then fill out the table with a row for each square.



n Number of pins along one edge	e Number of pins around the square	t Total number of pins on or inside square	i Number of pins in the interior
2	4	4	0
3	8	9	1
4			
5			
6			
7			
20			
50			
100			

For rows too big to set up on a geo board, you will need to find rules.

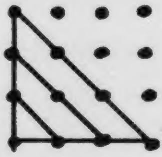
a) What is the rule to go from column n to column e? Once you've found it can you easily reverse e to n?

b) What is the rule from n to t?

c) Carefully compare t and i. Then can you change your rule for n to t so you can get a rule for n to i.

d) Find a rule to go from e and t to i.

2. As in problem 1, set up a sequence of right isosceles triangles at one corner of the geo board. Make enough so you can fill out enough of the table so you can figure out the rules and fill the blanks for triangles too big for the board.



No. of pins along one edge	No. of pins all the way around the triangle edges	Total number of pins on or in whole triangle	No. of pins in the interior of triangle
1			
2			
3			
4			
5			
6			
7			
20			
50			
	297		
		1770	
			13520

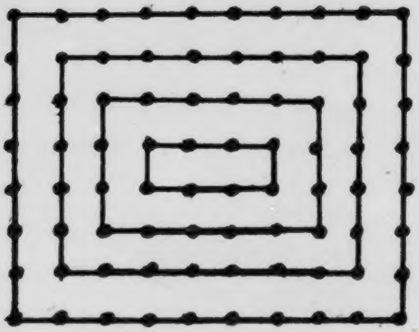
Write out the rules you have found.

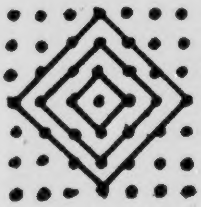
n to e: e =

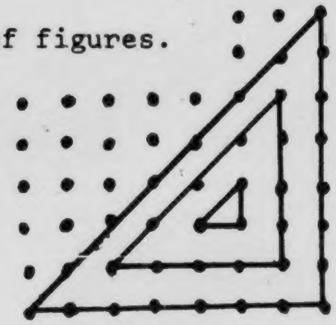
n to t: t =

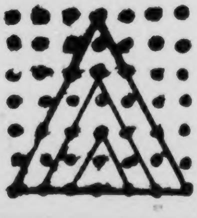
n to i: i =

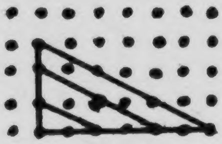
3. Make up similar problems with other sequences of figures.











II. ARITHMETIC PROGRESSIONS

Exercises

1. a) Draw a three sided figure. Find the sum of all the angles.

   b) Draw a four sided figure. Find the sum of all of its' angles.

   c) Draw a five sided figure. Find the sum of all its' angles.

   d) Draw a six sided figure. Find the sum of all of its' angles.

Now fill in the following table:

No. of sides	Sum of the angles
3	____
4	____
5	____
6	____
10	____
40	____
131	____
____	8460°
n	____

2. a. Look at the following pattern.

1 = 1
1 + 3 = 4
1 + 3 + 5 = 9
1 + 3 + 5 + 7 = 16

a) Add three more lines to the table.

b) Can you express the right-hand column differently?

b. How many odd numbers must be added to find 1296?

c. What is the sum of the first 100 odd counting numbers?

d. What is the fifth odd counting number?

i) What is the one-hundredth odd counting number? Do you detect a pattern?

ii) What is the "kth" odd counting number?

e. What is the sum of the first "n" odd counting numbers?

i) Can you generalize your discovery in words?

ii) Can you write a formula for finding the sum of the first "n" odd counting numbers?

f. How many terms would it take so that the sum will be close to 115? (It might not be exactly 115, but make it as close as possible.)

g. What is the sum of 21 + 23 + 25 + ... + 137 + 139?

3. Write in the missing numbers in the following arithmetic sequences and name the common difference of each.

a) 3, 5, 7, 9, ____, ____, ____. d = ________

b) 11, 15, ____, 23, ____, ____. d = ________

c) ____, ____, 23, 32, 41 d = ________

d) 5, ____, ____, ____. d = ________

e) -14, -10, ____, ____, ____, ____, 10 d = ________

f) 10, 7, 4, ____, ____, ____. d = ________

4. Write down the first four terms of the sequences whose general terms are given below. (n represents a natural number)

a) $2n$ c) 2^n e) $3-2^{n-3}$

b) $5n-3$ d) $\frac{1}{n}$

5. Write the first five terms of each of the following arithmetic progressions with first term a and common difference d.

a) $a = 3$, $d = 4$ c) $a = \frac{4}{5}$, $d = 7$ e) $a = \frac{3}{4}$, $d = \frac{1}{5}$

b) $a = -1$, $d = 5$ d) $a = 3$, $d = -4$

6. From the following list of sequences, select all of the arithmetic progressions and name the common difference of each.

1. 2, 5, 8, 11, ...
2. $\frac{4}{9}, \frac{10}{9}, \frac{16}{9}, \frac{22}{9}, \ldots$
3. 4, 8, 16, 32, ...
4. 2, 9, 16, 23, ...
5. 4, 4, 4, 4, ...
6. 10, 7, 4, 1, -2, ...
7. $\frac{2}{5}, \frac{4}{15}, \frac{8}{45}, \frac{16}{135}, \ldots$

7. Name the designated term, the n^{th} term and state which term the 2^{nd} designated number is:

a) 1,5,9,13,... 139th term is __ nth term is __ 725 is the __ term

b) -4,-1,2,5,... 100th term is __ nth term is __ 461 is the __ term

c) $\frac{2}{3},\frac{4}{3},\frac{6}{3},\frac{8}{3},\ldots$ 51st term is __ nth term is __ $\frac{128}{3}$ is the __ term

d) 7,9,11,13,... 213th term is __ nth term is __ 313 is the __ term

e) 3,10,17,24,... 47th term is __ nth term is __ 962 is the __ term

f) 16,10,4,-2,... 81st term is __ nth term is __ -584 is the __ term

8. Find the sum of the following arithmetic progressions.

a) 3, 6, 9, ... to 153 terms.

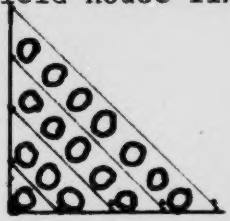
b) -5, -1, 3, 7, ... to 37 terms.

c) $-7 + (-4) + (-1) + 2 + \ldots + 191.$

d) $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{2}{3} + \ldots + 8\frac{2}{3}$

e) $253 + 250 + 247 + \ldots + 10$

9. A baseball club keeps its bats standing on end in diagonal rows in a corner of the field house like this:



a) How many bats would there lie in the 15th row?
b) How many in the 30th row?
c) What row would contain 115 bats?
d) If there are 30 rows of bats, how many bats in all are in the corner?
e) How many rows of bats would be necessary to have 16,900 bats on hand for the children on bat day?

III. GEOMETRIC PROGRESSIONS

Exercises

1. From the following geometric progressions name the common ratios.

   a) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots$

   b) 4, 8, 16, 32, ...

   c) 5, 5, 5, 5, ...

   d) $\frac{2}{3}, \frac{6}{9}, \frac{18}{27}, \frac{54}{81}, \ldots$

   e) $\frac{2}{5}, \frac{4}{15}, \frac{8}{45}, \frac{16}{135}, \ldots$

2. Write the 1st n terms of the geometric progressions with the following information.

   a) $a = 3$, $r = 2$, $n = 6$

   b) $a = -2$, $r = 3$, $n = 5$

   c) $a = 8$, second term 4; $n = 7$

   d) $a = \frac{-1}{4}$, fourth term 2, $n = 5$

   e) $a = -1$, $r = -3$, $n = 8$

3. Write the first ten terms of the following geometric porgressions, when the first term and common ratios are, respectively:

   a) 3, 4

   b) 10, -2

   c) 5, -1

   d) 1, 2

   e) 3, $\frac{1}{2}$

4. Name at least three more terms of the following sequences.

   a) 1, 3, 9, 27, ...

   b) 4, -4, 4, -4, ...

   c) $\sqrt{3}, \sqrt{6}, 2\sqrt{3}, 2\sqrt{6}, \ldots$

   d) 3, -6, 12, -24, 48, ...

   e) 3, 6, 12, 24, ...

5. Write the first five terms of the geometric porgression whose fifth term is 4 and whose common ratio is -2.

6. Find the second and third terms of a geometric progression whose fourth term is 81 and whose eighth term is 16.

7. Find the sum of the terms of the geometric progression described in each of the following:

   a) $a = 1, r = 2, n = 5$

   b) $a = 12, r = 2, n = 7$

   c) $a = \frac{1}{2}, r = 2, n = 8$

   d) $a = \frac{1}{25}, r = 5, l = 125$

   e) $a = 625, r = .2, n = 5$

8. Ten men are fishing from a pier. The first is worth $3,000; the second, $9,000; the third, $27,000 and so on. How many millionaires are in the group?

9. If a woman deposits $200 at the beginning of each year in a bank that pays 5% compounded annually, how much will her investment be worth at the end of seven years?

10. A man left one-third of his estate to his eldest child, one-third of the remainder to the next to the oldest, one-third of what then remained to the next, one-third of the amount remaining to the fourth, and the remaining $10,000 to his wife. Find the value of the estate.

11. Consider the decimal expansion of 4/7 which begins this way:

    .5714285.....

    a) Can you figure out the 30th digit in the above expansion?

    b) Find at what places in the expansion 5 appears.

    c) Can you find a way of telling what the nth digit in the expansion is for any n?

12. Express .4444... as a geometric series. Use this to find a common fraction which is the same as .4444....

IV. HARMONIC SEQUENCES

Exercises

1. Consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

a) What is the nth term of the above sequence?

b) How many terms will you have to add for the sum of the series to be slightly greater than 2?

c) How many terms will you have to add for the sum to become slightly greater than 5?

d) Now consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \ldots$$

How many terms will you have to add for the sum to become equal or slightly more than 5?

e) Compare the series in a) and d).

2. The Harmonic mean of two numbers is defined to be a number X, such that

$$\frac{1}{X} = \frac{\frac{1}{a} + \frac{1}{b}}{2}$$

a) Using the above definition, find the harmonic mean between 2, 4.

b) Prove that $X = \frac{2ab}{a + b}$

Consider the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$$

c) Continue this sequence to 10 terms.

d) What is the 20th term for the above sequence?

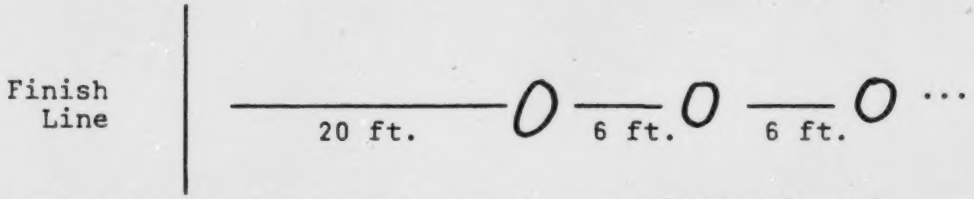
e) What is the nth term for the above sequence?

EXERCISE SHEET ON VARIOUS KINDS OF SEQUENCES

1.

1st 4 Terms	7th Term	10th Term	439th Term	Which Term?	nth Term	Kind of Sequences
4,6,8,10,...				182 is the ___ term		
-1,-2,-3,-4,...				-293 is the ___ term		
				191 is the ___ term	4n-1	
2,4,8,16,...		1024	2439	4096 is the ___ term		
3,6,3,6,...				6 is the ___ term		
				170 is the ___ term	n^2+1	
3,6,12,24,...				3072 is the ___ term		
				40 is the ___ term		
1,4,9,16,...				256 is the ___ term		
				28 is the ___ term	$\frac{n(n+1)}{2}$	
				48 is the ___ term	n^2-1	
5,5,5,5,...				5 is the ___ term		
1,4,7,10,...				934 is the ___ term		

2. In a potato race, 13 potatoes are placed 6 feet apart in a straight line. If the potatoes are to be picked up and returned to the finish line one at a time and if the first potato is 20 feet from the finish line, what distance will be covered by a contestant who finishes the race?



3. In a certain colony of bacteria each divides into 2 every hour. How many will be produced from a single bacillus if the rate of division continues for 12 hours?

4. A rubber ball is dropped from a height of 81'. Each time it strikes the ground it rebounds 2/3 of the distance through which it last fell. What distance has the ball traveled when it strikes the 6th time?

5. A charity raffle consists of 60 tickets numbered consecutively in multiples of 5 from 5 to 300, inclusively. If each purchaser pays in pennies the amount of the ticket he draws, how much will be taken in from the raffle?

6. If 1, 4, and 19 are added to the first, second and third terms, respectively of an arithmetic progression with $d = 3$, the first terms of a geometric progression obtained. Find the arithmetic progression and the common ratio of the geometric progression.

V. MATHEMATICAL INDUCTION

1. Find the sum of the first n odd numbers using the method of mathematical induction.

   a) In the first column at the bottom cover the first odd number of pins.

   b) Put another rubber band on the geo-board so that it covers the second odd number of uncovered pins closest to the pin already covered. How many pins have you covered so far?

   c) Now put another rubber band on the geo-board so that it covers the third odd number of uncovered pins which are closest to the pins already covered. What is the total number of pins now covered? What does this number represent?

   d) Can you use the geo-board to find the sum of the first four odd numbers? First five? Can you guess the sum of the first ten odd numbers? Can you guess a formula which will give the sum of the first n odd numbers? Check your formula for several values of n.

   e) If we go from the sum of the first five odd numbers to the sum of the first six, how many pins do we have to add? How many do we have to add in going from the the sum of the first six to the sum of the first seven? Seven to eight?

   f) What does your formula give for the sum of the first k odd numbers? What do we have to add to this to get the sum of the first k + 1 odd numbers? What is the sum of the first k + 1 odd numbers? Does this agree with what your formula says the sum of the first k + 1 odd numbers is?

   g) Do you have any conclusions about your formula?

2. Using the method of mathematical induction find:

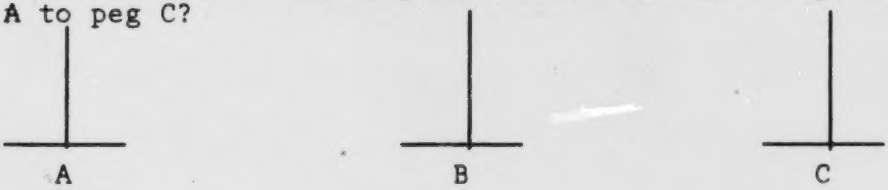
   a) the sum of the first n even numbers

   b) the sum of the first n natural numbers

Exercises

1. Using a regular Tower of Hanoi puzzle, what is the minimum number of moves it takes to move 7 disks from one peg to another? You may only move one disk at a time and may never put a larger one on a smaller one.

HINT: Start with 1 disk. How many moves will it take to go from peg A to peg C?



Do the same as above with 2, 3, and 4 discs.

Can you guess a formula that will relate the number of moves to the number of discs. Prove it by using mathematical induction.

a) Prove that $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$, using mathematical induction.

b) Prove using mathematical induction, that $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = a(1 - r^n)/(1 - r)$.

c) Prove, using mathematical induction, that $4 + 3 + 2 + 1 + \ldots + (5 - n) = 1/2n\,(9 - n)$.

d) Can you use exercise 11 to guess a formula for $x^n - y^n$? Prove it by mathematical induction.

e) Find a formula which tells the number of non-zero numbers having at most n digits in base 2.

2. Prove that the nth odd number is (2n + 1) using the method of mathematical induction.

First take a geo-board and rubber bands.

a) In the first column at the bottom of the geo-board put a rubber band around the first odd number of pins.

b) In the second column at the bottom put a rubber band around the second odd number of pins, and continue this in the third and fourth columns.

c) If this process is continued, how many pins will you cover in the fifth column? Sixth column? Can you guess how many would be covered in the tenth column? How about the nth column? Can you guess a formula or expression which will give the nth odd number?

d) Does your formula correctly give the first odd number? The second? The fourth? When you went from the fourth column to the fifth, how many more pins did you have to cover? Is your formula correct for the sixth odd number?

e) What is the 15th odd number? How do you get from the 15th to the 16th odd number? Is this what your formula gives for the 16th odd number? What does your formula give for the kth odd number? How do you get from the kth to the (k + 1)st odd number? Is this what your formula gives for the (k + 1)st odd number?

f) What can you conclude about your formula?

Go through a similar investigation with the even numbers. Find a formula for the nth even number and test the formula in the manner above.

Investigate the multiples of three similarly.

3. Prove $x^n - 1$ is always divisible by $x - 1$.

   a) What are the factors of the polynomial $x^2 - 1$?

   b) What can you do to the left side of the latter equation so that it becomes $x^3 - 1$?

   c) How do these same operations affect the right side of the equation?

   d) Can you factor this last expression? Can you now write the factors of $x^3 - 1$?

   e) Knowing the factors of $x^3 - 1$, can you factor $x^4 - 1$?

      HINT: How did you get the factors of $x^3 - 1$, knowing them for $x^2 - 1$?

   f) What are the factors of $x^5 - 1$? Can you guess what the factors of $x^n - 1$ are?

   g) If you knew the factors of $x^k - 1$, could you write the factors of $x^{k+1} - 1$?

   h) Use mathematical induction to prove that your formula for the factors of $x^n - 1$ is correct for all n.

VI. MANIPULATIONS WITH EXPONENTS

EXPONENT CHART

	0	1	2	3	4	5	6	7	8	9
2	1	2	4	8	16	32	64	128	256	512
3	1	3	9	27	81	243	729	2187	6561	19,683
5	1	5	25	125	625	3125	15,625	78,125	390,625	1,953,125

	10	11	12	13	14
2	1024	2048	4096	8192	16,384
3	59,049	177,147	531,441	1,594,323	4,782,969
5	9,765,625	48, 828,125	244,140,625	1,220,703,125	6,103,515,625

	15	16	17	18
2	32,768	65,536	131,072	262,144
3	14,348,907	43,046,721	129,140,163	387,420,489
5	30,517,578,125	152,587,890,625	762,939,453,125	3,814,687,265,625

	19	20	21	
2	524,288	1,048,576	2,097,152	
3	1,162,261,467	3,486,784,401	10,460,353,203	
5	19,073,436,328,125	95,367,181,640,625		

	22	23
2	4,194,304	8,388,608
3	31,381,059,609	94,143,178,827
5		

Exercises

In the preceding exponent chart, the powers of 2, 3, and 5 are given from the 0th power to the 23rd power. For example, from the chart we read:

$3^{16} = 43,046,721$　　$5^{8} = 390,625$　　$2^{14} = 16,384$

1. Perform the following multiplications:

$125 \times 625 =$ ____________

$16 \times 64 =$ ____________

$9 \times 243 =$ ____________

$5 \times 390,625 =$ ____________

Now examine the exponent chart and try to figure a way of getting these answers from the chart without actually computing them.

2. Using the chart as a guide, make up other more complicated multiplication problems and write their solutions without computing them.

3. Could you use the exponent chart to calculate 1024 x 177,147? Why?

4. Write a general rule which explains how you used the chart to get your answer. Does this rule hold for exponents in general? Explain.

5. Can you figure out how to use the exponent chart to solve the following problems?

$\frac{256}{4} =$ ____________

$\frac{59,049}{2187} =$ ____________

$\frac{390,625}{78,125} =$ ____________

What is the general rule in this case?

6. Make up other complicated division problems using the chart as a guide.

EXERCISE SHEET FOR LOGARITHMS

1. In the expression b^x, b is called the base, x is called the exponent, exponent, the whole b^x is the power, here the x^{th} power of b. If $b^x = p$ then p is also the x^{th} power of b.

Fill in as much of this table as you can.

power equation	exponent	power
$4^0 = 1$	0	1
$4^{.2} = 4^{\frac{1}{5}} = 1.32$		
$4^{.25} = 4^{\frac{1}{4}} = 1.41$		
$4^{\frac{1}{3}} = 1.59$		
$4^{\frac{1}{2}} =$		
$4^{\frac{2}{3}} =$	.67	
$4^{.75} = 4^{\frac{3}{4}} = 2.80$	.75	
$= 4$		4
$4^{1\frac{1}{3}} = 4^{\frac{4}{3}} =$		
$4^{1.5} = 4^{?} =$		
$4^{1\frac{2}{3}} = 4^{?} = 10.1$		
$4^2 = 16$	2	16

power equation	exponent	power
$4^{\frac{7}{3}} = 25.4$	2.33	
$= 30$		
$4^{\frac{8}{3}} = 40.3$		
$= 50$		50
$= 64$		
$4^{3.1} =$		
$4^{3.2} = 84.4$	3.2	
$= 90$		
$4^{\frac{10}{3}} = 101.6$		
$= 128$		

Let us see if there is any way we can find the missing powers. For $4^{\frac{5}{2}}$ it may seem strange to talk about miltiplying 4 by itself $\frac{5}{2}$ times. But study this: since $4 = 2^2$ then $(4)^{\frac{5}{2}} = (2^2)^{\frac{5}{2}} = 2^5 = 32$. This shows it is possible to talk about fractional exponents. See if you can fill in more of the table.

Now, $4 = 2^2$ translates into $\sqrt[2]{4} = 2$. In the same way $3^2 = 9$ translates into _____________, and $\sqrt[x]{p} = b$ translates into $b^x = p$. So, the identify $\sqrt[x]{p} = \sqrt[x]{p}$ translates into $(\sqrt[x]{p})^x = p$. This last is gotten by treating the $\sqrt[x]{p}$ just like the b above it. Since $(\sqrt[x]{p})^x = p$, then $a = (\sqrt[y]{p})^y$. Therefore, $(a)^{\frac{x}{y}} = ((\sqrt[y]{a})^y)^{\frac{x}{y}} = (\sqrt[y]{a})^{y \cdot \frac{x}{y}} = (\sqrt[y]{a})^x$ i.e., $a^{\frac{x}{y}} = (\sqrt[y]{a})^x$. This makes any fractional exponent make sense wherever roots and powers make sense.

This means $b^{\frac{3}{4}} = (\sqrt[4]{\quad ? \quad})^{?}$

$16^{\frac{3}{4}} = (\sqrt[?]{\quad ? \quad})^{?} = \underline{\qquad\qquad}$

$64^{\frac{2}{3}} = \underline{\qquad\qquad} = \underline{\qquad\qquad}$

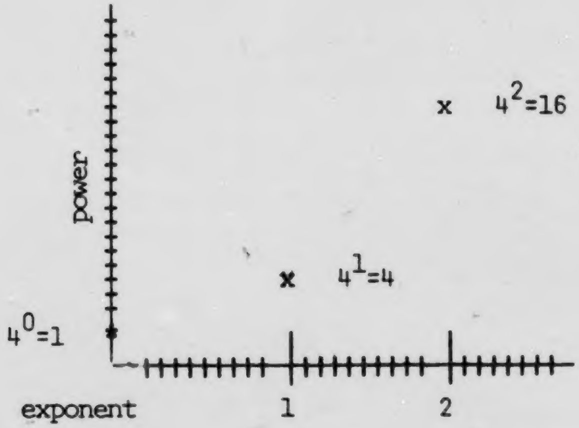
This gives a method of f-lling in some more of the powers in our table if we know:

$\sqrt[3]{4} = 1.59$

$\sqrt[4]{4} = 1.41$

$\sqrt[5]{4} = 1.32$

Now, let's make a picture graph of the information we have and then connect the points so we can find any power of 4. To do this accurately you will need a full piece of graph paper preferably metric that is 10 squares to a centimeter.



What is 4^3? Where will you plot the corresponding point? Now, plot the points that correspond to 4^0, 4 , 4^1, $4^{\frac{3}{2}}$, 4^2, $4^{\frac{4}{2}}$, etc., as well as the other powers you know from the table.

Now connect the points on your graph and then use it to answer the following:

1. $4^{.3} =$ __________
2. $4^{.7} =$ __________
3. $4^{x} = 21$ $x =$ __________
4. $4^{x.7} =$ ________
5. $4^{x} = 35$ $x =$ ________
6. $4^{.1} =$ ________
7. $4^{3.6} =$ ________
8. $4^{x} = 97$ $x =$ ________
9. $4^{1.3} =$ ________
10. $4^{x} = 54$ $x =$ ________

As you used the exponent tables, now use your powers-of-4 graph to calculate the following: the first will give you clues.

$1.6 \times 25.4 = 4^{.33} \times 4^{2.33} = 4^{.33+2.33} = 4^{2.66} = 40.4$

$15.8 \times 5.3 =$

$36.5 \times 2.9 =$

$100.4 \div 73.6 =$

$85 \div 3.7 =$

$\sqrt[2]{95.8} =$

$\sqrt[3]{80} =$

$2.7^{4} =$

2. Logarithm Language

Since you made no use of the 4 (the base) in the above computations you will probably prefer to use a language where you need not write it as you compute.

1. $b^x = N$ which is read "base b with exponent x is the number N" can be translated into $\log_b N = x$ which is read "the logarithm of N to the base b is x." In other words, a logarithm is simply another name for an exponent but the order you state it in exponent language is different from the order in logarithm language. In a similar way word order must be changed when we change from "father of" to "son of" -- "John is the father of Will" becomes "Will is the son of John." Thus, in exponent form $6^2 = 36$ becomes the logarithm of 36 to the base ______ is ______.

2. $4^3 = 64$ informs us that the logarithm of 64 to the ______ 4 is ______.

3. In $4^3 = 64$, the 3 is the ______ of 4 and 3 is also the ______ of 64 to the base 4.

4. $5^2 = 25$ translates into the ______ of 25 to the base ______ is ______.

5. $5^2 = 25$ becomes the logarithm of 25 to the base 5 is 2. This fact may also be represented as $\log_5 25 =$ ______.

6. The expression $\log_5 25 = 2$ states that the logarithm of ______ to the base 5 is ______.

Exponential Form	Logarithmic Form
$5^2 = 25$	$\log_5 25 = 2$

7. Both of the above statements say that the logarithm of ______ to the base ______ is ______.

8. The logarithm of 25 to the base 5 is 2. Note that the 2 is the ______ exponential form and the ______ in the logarithm form.

Fill each space with the appropriate word or number.

9. Since $2^7 = 128$, $\log_2 128 =$ ______.

10. Since $36^{1/2} = 6$, $\log_{36} 6 =$ ______.

11. In the statement $3^5 = 243$, the number 5 is the ______ of 3 and the ______ of 243.

12. $\log_7 49 = 2$ means $49 =$ ______.

13. $\log_3 27 = 3$ means $27 =$ ______.

14. If $\log_4 N = 5$ then $N =$ ______.

15. If $\log_5 125 = x$ then $5^x = 125$. Therefore, $x =$ ______.

16. If $\log_4 16 = a$ then $a =$ ______.

17. $\log_2 128 = x$; $x =$ ______.

18. $\log_b 64 = 2$; $b =$ ______.

19. $\log_3 N = 15$; $N =$ ______.

20. $\log_4 49 =$ ______.

21. $\log_4 N = .3$; $N =$ ______.

22. $\log_4 35 =$ ______.

23. $\log_4 N = 3.6$; $N =$ ______.

24. $\log_4 97 =$ ______.

25. $\log_{10} 10{,}000{,}000 =$ ______.

3. Now convert your base 4 power curve to a base 10 power curve. If the power is 100, what would the exponent of 10 be, i.e., if $10^x = 100$ then $x =$ ______. Now, $4^{3.32} = 100$ so re-mark the 3.32 on the horizontal exponent line with "2".

Next, since $\sqrt{10} = 3.16$, $10^{.5} =$ _____. $4^{1.66} = 10$, so $4^x = \sqrt{10}$? What point on the exponent line should be re-marked with ".5"? Could you find it without knowing $4^{1.66} = 10$?

D. Characteristics

What is $10^{1.5}$? Since $1.5 = 1 + .5$,

$10^{1.5} = 10^1 \times 10^{.5} = 10 \times 3.16 = 31.6.$

$10^{2.5} = ?$

$10^{3.5} = ?$

Using $10^{.1}$ = 1.259 find the following:

$10^{1.1}$ = ________
$10^{2.1}$ = ________
$10^{7.1}$ = ________
10^{x} = 12590 x = ________
$10^{8.1}$ = ______
10^{x} = .1259 x = ________
10^{x} = .01259 x = ________
10^{x} = .0001259 x = ________

When the power is changed by a multiple of 10, how is the exponent changed?

Fill in the following base 10 exponent chart:

power	1.259	1.585	1.995	2.51	3.16	3.98	5.01	6.31	7.94
exponent	.1	.2	.3	.4	.5	.6	.7	.8	.9

power	12.59		398			5010			
exponent		3.4		-1	-3		9.2	.2-4	.4-7

In many textbooks you can find a table similar to this. However, instead of giving the complete exponent, you'll only find the decimal part called the mantissa. State how you can determine the characteristic i.e., the integral part. The table is called a base 10 logarithm table.

TABLE I: FOUR-PLACE LOGARITHMS

Examples Illustrating Use of Tables:

1. Log 15 = 1.1761
2. Log 6.56 = 0.8169
3. Log N = 2.4771
   N = 300
4. Log N = 9.9786 - 10
   N = 0.952

n	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	16	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
n	0	1	2	3	4	5	6	7	8	9

TABLE I: FOUR-PLACE LOGARITHMS (Continued)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

VII. A MACHINE THAT ADDS

1. Use your paper adding machine to find the sums indicated below:

    a) 3 + 4

    b) 5 + 2

    c) 2 + 6

    d) 2.1 + 3.5

    e) 2.9 + 1.3

    f) 7.3 + 0.5

    g) 2.25 + 4.50

    h) 4.75 + 3.45

    i) 1.25 + 245

    j) 22 + 19

    k) 125 + 245

    l) 2.4 + 3.5

    m) 0.24 + 0.35

    n) 24 + 35

    o) 240 + 350

    p) 240,000 + 350,000

    q) 8 + 4

    r) 7.2 + 5.9

2. Did you have trouble finding any of the sums in #1? If so, what went wrong? How does the decimal point affect the method you used for finding sums?

3. Can you use your "machine" to calculate 320 + 26.5?

4. Why do you have to be particularly careful if you want to add 11.6 and 5.8?

5. Could your "machine" give an answer like 12.45 or 23.58? Why?

6. Can you make your "machine" subtract? Explain.

7. Use your machine to perform the calculation indicated below:

    a) 8 - 3

    b) 7 - 2

    c) 12 - 5

    d) 23 - 8

    e) 48 - 26

    f) 330 - 185

    g) 5.8 - 3.9

    h) 7.4 - 0.8

    i) 0.69 - 0.44

    j) 1 - 0.05

    k) 12.5 - 6.5

    l) 245 + 615 - 480

    m) 1200 + 4100 + 8500

8. a) Where is 0 on your scale?

   b) Where is 1?

   c) How far from 0 is 7? 8? (3+5)? (4+2)?

   d) Is it essential to use a scale divided into equal units to make your machine add?

   e) What do you think will happen if 2 is not the same distance from 3 and 3 is from 4? Try it.

9. Design a machine which will add fractions with denominators 2, 3, 4, 6 and 12. Demonstrate that your machine works with 3 examples using different denominators.

VIII. A SPECIAL SLIDE RULE

1. Make an adding machine with scales from 0.00 to 1.00. Locate and lightly mark the points 0.0, 0.1, 0.2, 0.3, ..., 0.9, and 1.0.

2. As accurately as possible, locate and clearly mark the points .00, .30, .48, .60, .70, .78, .85, .90, .95, .99, and 1.00. We will call this group of markings the 'regular' number scale.

3. Label the same points respectively with the following 'alternative' marks at .00 also put 1, at .30 also put 2, at .48 also put 3, at .60 put 4, at .70 put 5, at .78 put 6, at .85 put 7, at .90 put 8, at .95 put 9, at .99 put 9.8, and at 1.00 put 10.

4. Use this adding machine to do the following additions using the 'regular' scale. Then below each number write the 'alternative' scale readings at the same marks. (Estimate if necessary.)

a) .30 + .48 = ________

____ ____ = ________

b) .30 + .30 = ________

____ ____ = ________

c) .48 + .48 = ________

____ ____ = ________

d) .70 + .30 = ________

____ ____ = ________

e) .38 + .30 = ________

____ ____ = ________

f) .23 + .30 = ________

____ ____ = ________

g) .12 + .85 = ________

____ ____ = ________

h) .38 + .60 = ________

____ ____ = ________

5. For the 'alternative' scale values in problem 4, what symbol between the numbers on the left of '=' will make the equation true?

6. As in problem 4, use your adding machine's regular scale to perform the following subtractions:

a) 0.96 - 0.48 = ________

____ ____ = ________

b) 0.60 - 0.30 = ________

____ ____ = ________

c) 0.78 - 0.30 = ________

____ ____ = ________

d) 1.00 - 0.30 = ________

____ ____ = ________

e) 0.68 - 0.38 = ________

____ ____ = ________

f) 0.53 - .30 = ________

____ ____ = ________

g) 0.97 - 0.12 = ________

____ ____ = ________

h) 0.98 - 0.60 = ________

____ ____ = ________

7. For the corresponding alternative scale values of problem 6, what operation symbol between the numbers on the left will make the equations true?

8. Use the logarithm table to complete the following statements as in this example. log 4.8 = 0.68.

a) log 2 = ________

b) log ____ = .60

c) log ____ = 0.20

d) log 3 = ________

9. Use a logarithm table to see how the numbers on the regular and alternative scales of your machine are related. Decide for each of the following statements whether it is true or false. (numbering refers to problem 4 or 6)

4 a) log 2 + log 3 = log (2 x 3)

4 b) log 0.30 x log 0.30 = log 0.60

4 c) log 3 + log 3 = 0.96

4 d) log 5 + log 2 = log 10

6 a) log 9 - log 3 = log (9 ÷ 3)

6 b) log 4 - log 2 = 0.30

6 c) 6 - 2 = log 3

6 d) log 10 - log 2 = 0.70

6 e) log 4.8 - log 2.4 = log (4.8 ÷ 2.4)

10. Referring to problems 4 or 6, translate the following problems into logarithm language as is done in problem 9 in 4 a) or 6 a).

a) 4 d) ______________

4 e) ______________

b) 6 d) ______________

6 e) ______________

IX. TEST YOURSELF

A. SEQUENCES

Find the following sums:

1. $2 + 2 + 2 + 2 + 2 + 2$

2. $-5 + (-1) + 3 + 7 + \ldots + 1067$

3. $\frac{1}{5} + \frac{3}{5} + \frac{9}{5} + \frac{27}{5} + \ldots + \frac{1{,}594{,}323}{5}$

4. $\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 + 2 + \ldots + 1024$

5. $\frac{1}{3} + \frac{11}{15} + 1\frac{2}{15} + 1\frac{3}{15} + 1\frac{14}{15} + \ldots + 19\,\frac{14}{15}$

Find the 153rd and nth terms of the following:

6. $-3, 1, 5, 9, \ldots$

7. $2, 6, 18, 54, \ldots$

8. $0, 3, 8, 15, 24, 35, \ldots$

9. $2, 2, 2, 2, 2, \ldots$

10. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Name the common difference or the common ratio, whichever is applicable of each of the following:

11. $\frac{2}{5}, \frac{6}{25}, \frac{18}{125}, \frac{54}{625}, \ldots$

12. $3, 3, 3, 3, \ldots$

13. $-10, -6, -2, 2, \ldots$

14. $3, 1, -1, -3, -5, \ldots$

15. $4, -4, 4, -4, \ldots$

16. $3, \frac{-3}{5}, \frac{3}{25}, \frac{-3}{125}, \ldots$

17. $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, \ldots$

18. $2, 5, 8, 11, \ldots$

19. $7, 17, 27, 37, \ldots$

20. $3, 6, 12, 24, \ldots$

B. LOGARITHMS

I. Find the following:

1. $4^2 = 16$, $\log_4 16 =$ _____

2. $2^3 = 8$, $\log_2 8 =$ _____

3. 8^2 = _____, $\log_8 64 = 2$

4. $\log_9 49$ = _____

5. $\log_4 2$ = _____

II. Find the following logarithms of these various powers of 2.

6. $\log_2(16 \times 8)$ = _____

7. $\log_2 \left(\frac{16}{8}\right)$ = _____

8. $\log_2(8{,}192 \times 128)$ = _____

9. $\log_2 \left[\frac{8 \times 16 \times 32 \times 64}{128}\right]$ = _____

III. Fill each blank in in problems below with the appropriate number or expression.

10. $\log_6 \sqrt[3]{458}$ = _____ x $\log_6 458$

11. $\log_8 (795^4 \times 816^6)$ = _____ $\log_8$ _____ + _____ $\log_8$ _____

12. $\log_4 \frac{\sqrt[6]{917} \times 314 \times 21^2}{\sqrt[6]{412} \cdot \sqrt[5]{16}}$

Given: $\log_{10} 2 = .3010$; $\log_{10} 3 = .4771$, $\log_{10} 5 = .6990$, find:

13. $\log_{10} 40$ = _____

14. $\log_{10}(3\sqrt{5} \times 8)$ = _____

15. $\log_{10} \left(\frac{5}{3}\right)$ = _____

16. $\log_{10} 15{,}000$ _____

Find logarithms of the following numbers:

17. log 869 = _____

18. log 0.00071 = _____

19. log 762,000,000,000 = _____

20. log 214.2 = _____

Find the antilogarithm of the following numbers:

21. log N = 3,6160, N = _____

22. log N = 0.4014, N = _____

23. log N = 6.8949-10, N = _____

Compute the following using logarithm:

24. $\sqrt{76.3}$

25. $(49.7)^5$

26. (.00529)(721)

27. $\dfrac{\sqrt{786}}{(21.2)^3 \times 8.13}$

C. SLIDE RULE

1. 2 x 3 =

2. 20 x 300 =

3. 7.2 x 850 =

4. 47.3 x 3.9 =

5. 6850 x .0584 =

6. $\dfrac{1}{728} =$

7. $\dfrac{1}{.0297}$

8. 67.3 ÷ 8.48

9. $\dfrac{.749}{023.4}$

10. $\dfrac{4.84 \times 8950}{26.2 \times 366}$

ANSWERS

A. SEQUENCES

1. $6 \times 2 = 12$

2. $\frac{(1067-5)269}{2} = \frac{1062}{2} \times 269 = 531 \times 269 = 142,839$

3. $\frac{4,782,969-1}{2 \times 5} = 478,296.8$

4. $2048 - \frac{1}{16} = 2047\ \frac{15}{16}$

5. $19\ \frac{14}{15}$ is 50th term $(19\ \frac{14}{15} + \frac{1}{3}) \times 25 = \frac{299 + 5}{15} \times 25 = \frac{304}{3} \times 5 =$

$\frac{1520}{3} = 506\ \frac{2}{3}$

	153rd term	nth term
6.	605	$4n-7$
7.	2.3^{152}	2.3^{n-1}
8.	23,408	n^2-1
9.	2	2
10.	$\frac{1}{2153} = 2^{-153}$	$\frac{1}{2n} = 2^{-n}$

11. $r = \frac{3}{5}$
12. $d = 0$, $r = 1$
13. $d = 4$
14. $d = -2$
15. $r = -1$
16. $r = 2$
17. $r = 2$
18. $d = 3$
19. $d = 10$
20. $r = 2$

B. LOGARITHM

I.
1. 2
2. 3
3. 64
4. 2
5. $\frac{1}{2}$

II.
6. 7
7. 1
8. 20
9. 11

III. 13. 1.6020

14. 1.1360

15. .2219

16. 4.1761

V. 17. 2.9390

18. 6.8513-10

19. 11.8820

20. 2.3308

C. SLIDE RULE

1. 6

2. 6000

3. 6120 (for the decimal think: 7 x 800 = 5600)

4. 184 or 185 (for decimal, 40 x 4 = 160)

5. 400 (for decimal, 7000 x .06 = 70 x 6 = 420)

6. .00137 (for decimal $\frac{1}{1000} = .001$

7. 33.7 (for decimal, $\frac{1}{.02} - \frac{100}{2} = 50$)

8. 7.94 (for decimal, 64 ÷ 8 = 8)

9. .0320 (for decimal, $\frac{.75}{25} = .03$) (The zero after the 2 shows 3 place accuracy)

10. 4.52 (for decimal, $\frac{5 \times 9000}{30 \times 400} = \frac{45\cancel{0}\cancel{0}\cancel{0}}{12\cancel{0}\cancel{0}\cancel{0}} = 4$)

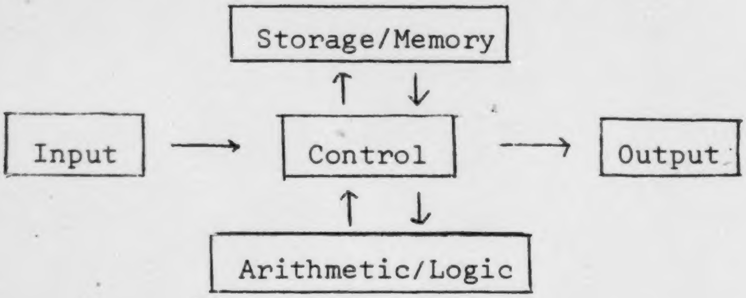
- COMPUTER-BASIC -

Computers are designed to be operated by a machine language -- a sequence of coded commands which are stored in the memory unit until called into use by the control unit. Early programmers found it necessary to learn this machine language which bears no resemblance to any spoken tongue. Synthetic languages are being developed which are designed to bridge the gap between the spoken tongue and the machine language.

The purpose of this unit is to get a feel for what is involved in writing a set of instructions so a computer can give the answer to some problem. The instructions given to the computer constitute a computer program. The language we will use is BASIC (Beginner's All-purpose Symbolic Instruction Code) designed for use with the Time-Sharing service.

HOW THE COMPUTER WORKS

A diagram of the flow of information through a computer roughly resembles the one below.



Input consists of the set of statements which make up the program. Input to the computer may be transmitted by punched cards, tape, teletype, or from the keyboard on the console. The statements may be instructions to the computer to perform certain operations or data to be used in the execution of the instructions.

The storage/memory unit is where information is stored until needed in the execution of some part of the program.

The arithmetic/logic unit is where the actual computations take place. This unit may add, subtract, multiply or divide; higher-order mathematics is not done directly. This unit also performs certain logical operations; it can determine if a number is positive or negative, zero or non-zero.

The control unit acts in a supervisory capacity and exercises control over all units in the system. It coordinates the activities of the other units by timing and directing the flow of information from one unit to another.

Output may be in the form of punched cards, punched tape, magnetic tape or printed sheets. The output consists of the results obtained from the computer operations.

I. GIVING INSTRUCTIONS

Exercises

1. Write a program to sort by suit. Also, arrange the cards in each box in ascending order.

2. Write a set of statements that will direct me to your home from here. (This assignment should lead to some very interesting discussions.) Two very important ideas should come out of these discussions:

   (a) The instructions must be given in a mutually understood language.

   (b) Directions (left, right, etc.), and street and highway names should be used for precision.

   Ask some of the students to read their instructions and ask:

   (a) Why did you write your statements in English and not Spanish or French, etc.?

   (b) Why did you list your statements? Why did you not write them in paragraph form?

   (c) Why were street and highway names necessary?

   (d) What must one know to be able to execute the instructions?

II. INSTRUCTIONS IN BASIC

Below is a sequence of arithmetic formula statements (Program) in the Basic Language to be executed in the order they are written. After each statement has been executed, what is the current value stored in each variable location assuming each was zero at the beginning of the program?

A	B	C	PROGRAM 1
_____	_____	_____	10 LET A = 5
_____	_____	_____	20 LET B = -A
_____	_____	_____	30 LET C = A/B - 1

```
  A        B        C              PROGRAM 1
______   ______   ______    40    LET C = C + 1
______   ______   ______    50    LET B = B * B + C
______   ______   ______    60    LET A = A ↑ 2 - B
______   ______   ______    70    LET B = B - C + A
______   ______   ______    80    LET C = B * C
______   ______   ______    90    LET B = B/2
______   ______   ______   100    LET A = C/B + 12
```

```
  A        B        C              PROGRAM 2
______   ______   ______    10    LET A = 5
______   ______   ______    20    LET B = A - A * A
______   ______   ______    30    LET C = B - B/A
______   ______   ______    40    LET A = B + A * C/B
______   ______   ______    50    LET C = B * (A + C)/A
______   ______   ______    60    LET A = A * C ↑2/B
______   ______   ______    70    LET A = A/C * B
______   ______   ______    80    LET B = C/A * B↑ (C/B)
______   ______   ______    90    LET A = B * A/C(C - 2 * B)↑ 2
______   ______   ______   100    LET C = C ↑ 2/2 * (C - 2 * B)
```

Exercises

1. If A = 12 and B = 4, write each of the following statements in Basic so that the computed value of each variable is as given:

45 1. $C = B \cdot A - A/B$

3 2. $D = \dfrac{C \cdot B - A}{A}$

6 3. $B = \dfrac{C \cdot B + A \cdot D}{A \cdot D}$

8 4. $A = \dfrac{A}{B} \cdot \dfrac{C + D}{A}$

2 5. $X = \frac{A(C + D^2))}{B^D}$

Run your program on the computer to see if your statements are correct.

2. Write five arithmetic statements that are acceptable to the computer. Then, put them on the computer to see if they are.

III. ENGLISH TO BASIC

1. Can you execute the following instructions?

Pick two numbers.

Double the first.

Add seven.

Add three times the second.

Subtract 11.

What do you get?

2. Can you write a statement in Basic that will correspond to each statement above?

3. Can you execute the following instructions?

Pick two numbers.

Triple the second.

Add eight.

Subtract the first.

Add seven.

What do you get?

4. Can you write a statement in Basic that will correspond to each statement above?

Exercises

1. Write a program to find the average of four numbers.
2. Write a program to find the sum of the first 5 odd numbers.
3. For any given counting number, write a program that will find three consecutive odd numbers.
4. Write a program that will find the next odd number greater than any given counting number. (The students should be told about the integer function -- Let Y = INT (X) for X = 5/2 assigns to Y the value 2 instead of 2.5.)

IV. CONDITIONAL CONTROL

After each statement in the following sequence has been executed, what is the statement number of the statement that the computer will execute next when X = 10 and Y = 20?

```
_____      10  IF ( X )< 5 THEN 50

_____      20  IF 2 * X = -Y THEN 70

_____      30  IF X + Y >5 THEN 80

_____      40  IF Y - X < =5 THEN 60

_____      50  IF 4 * X - Y > = 15 THEN 70

_____      60  IF X - 2 * Y < -20 THEN 50

_____      70  IF 3 * X - X >15 THEN 90

_____      80  IF 4 * Y/-X  < = -8 THEN 40

_____      90  IF Y - 2 * X >= 10 THEN 20

_____     100  IF 2 * X + Y <> 10 THEN 10
```

Exercises

1. What is the output for the following program? Run the program to check your answer.

```
                               I        N
10  LET N = 10               _____    _____

20  LET I = I + 3            _____    _____

30  IF I > N THEN 60         _____    _____
```

```
                           I       N
40 PRINT I, N            _____   _____

50  GO TO 20             _____   _____

60  END
```

2. Assume that the program shown below has been completely executed, how many times was each statement executed?

```
_____     10    LET N = 6

_____     20    LET I = I + 2

_____     30    IF N < I THEN 55

_____     40    LET N = N + 1

_____     50    GO TO 20

_____     55    LET L = N

_____     60    IF I - N > 10 THEN 75

_____     70    GO TO 20

_____     75    LET L = I

_____     80    END
```

Before running Program 2 above, the following statements should be added:

```
15 PRING "10"

25 PRINT "20"

28 PRINT "30"

45 PRINT "40"

48 PRINT "50"

57 PRINT "55"

58 PRINT "60"

65 PRINT "70"

77 PRINT "75"

78 PRINT "80"
```

By counting the number of times a statement number is printed, you can determine the number of times that statement was executed. You may also use this printout to retrace the steps of the computer in the program.

3. What is the output for the given data for the following program? Run the program to check your answer.

```
                                   M       K

10  LET K = 0                    _____   _____

16  LET J = 1                    _____   _____

20  READ M                       _____   _____

30  IF M = 0 THEN 60             _____   _____

40  LET K = M + J + K            _____   _____

50  GO TO 70                     _____   _____

60  LET K = K - 1

70  PRINT M, K

80  LET J = J + 1

90  IF J < = M THEN 20

98  END
```

4. Assume that the program shown below has been completely executed, how many times was each statement executed?

```
_____   10   LET I = 20

_____   20   LET N = I

_____   30   LET I = I

_____   35   IF 2 * I - N< 0 THEN 80

_____   40   LET H = SQR(N↑ 2 + I ↑2)

_____   60   LET N = N + I/2

_____   70   GO TO 30

_____   80   LET I = I + 10

_____   90   IF 100 - I >  0 THEN 20

_____   95   END
```

Since some of the statements in Program 4 on the previous page will be executed many times, you should add in counters instead of the PRINT statements as in Program 2 and PRINT statements before the END statement.

Add:

```
    15    LET A = A + 1

    25    LET B = B + 1

    32    LET  C = C + 1

    34    LET D = D + 1

    45    LET E = E + 1

    65    LET G = G + 1

    68    LET J = J + 1

    85    LET K = K + 1

    88    LET L = L + 1

    92    PRINT "10-"; A; ", 20-"; B; ", 30-"; C; ", 35-";
                D;", 40-" E

    94    PRINT "60-"; G;", 70-"; J;", 80-"; K;", 90-"; L;",
                95-1"
```

5. Given four variables, A, B, C and D with previously defined values, write a sequence of "if, then" statements that will instruct the computer to execute statement number 80, if all four variables have the same value. If one or more variables have different value, tell the computer to execute statement 45 instead.

6. Given three variables X, Y and Z with no two values the same, if the value of X is:

    (a) greater than both Y and Z, execute statement number 10 next,

    (b) less than Y but greater than Z, execute statement number 20 next,

    (c) greater than Y but less than Z, execute statement number 30 next, and

    (d) less than both Y and Z, execute statement number 40 next.

7. Write a program that will sum the first 20 counting numbers.

8. Write a program that will get the square root of the first 20 counting numbers.

9. Write a program that will find the first 20 odd positive integers.

10. Write a program that will find the prime numbers less than one hundred.

What is the printout of the following program? Run the program to check your answer.

```
10    LET N = 1
20    LET A(N) = N * (N + 1)/2
30    PRINT N, A(N)
40    LET N = N + 1
50    IF N <= 10 THEN 20
60    END
```

	N	A(N)
1st	_____	_____
2nd	_____	_____
3rd	_____	_____
4th	_____	_____
5th	_____	_____
6th	_____	_____
7th	_____	_____
8th	_____	_____
9th	_____	_____
10th	_____	_____

Exercises

1. Set each value of an array called ID equal to the order of the value in the array (that is, ID (1) = 1, ID (2) = 2, etc.). The array has length 20.
2. Store the elements of array A in array B.
3. Store the elements of array A in array B in reverse order.
4. Find the smallest number in an array of 10 numbers.
5. Write a program that will find the pythagorean numbers less than one hundred.
6. Write a program that will subdivide a line eight times.
7. Write a program that will find the number of divisors of a number.
8. Evaluate an N degree polynomial.
9. Arrange a sequence of numbers in ascending order.
10. Find the roots of a polynomial equation to three decimal places.

- PROBABILITY -

I. MULTIPLICATION PRINCIPLE

1. Given the set of digits 0, 1, 2, 3, 4, how many five digit odd numbers can you find? How many even numbers of five digits will there be? Can you answer the last question without using the answer from the first question? This problem is probably more interesting if repetitions are not allowed.

2. We use 10 digits, 0, 1. 2, 3, 4, 5, 6, 7, 8, 9, to write numbers. How many 3-digit numbers can we write whose first digit is odd, whose second digit is 5, and whose third digit can be anything if (a) repetition is allowed; (b) if it is not.

3. You are going to walk six blocks through the streets of a city, but you do not care where you go. At each intersection you can either turn left or right or go straight ahead. How many different paths could you follow?

4. The puzzle "Instant Insanity" is made up of 4 blocks whose faces have 4 different colors. Some are white, some red, some blue and some green. None of the blocks has the same arrangement of faces. The object of the puzzle is to stack the blocks in such a way as there are 4 different colors on each side.

   a) How many faces does a single cube have?

   b) In how many different ways can one cube be placed?

   c) How many different looking stacks can be made with 4 blocks?

5. If you toss one coin, how many different outcomes do you expect? __________ List them.

   Suppose you toss two coins (a penny and a dime, for example). How many different outcomes are possible? List them.

   How many outcomes do you think are possible if you toss 3 coins at once? ______

   Make a reasonable guess for the number of possible outcomes if you toss 7 coins at once.

On what did you base your guess?

Does your method work for the tossing of 4 coins? _____
Test it by actually listing all possible outcomes.
If your method did not work, look through the problem again.

However, if your method works, you should now be able to state the number of outcomes in the experiment of tossing 8 coins, 9 coins, 10 coins, etc. Try to generalize this using N coins.

6. Suppose instead of tossing a coin you were rolling a pair of standard dice. How many outcomes are possible? _____

   How many outcomes are possible in the experiment of rolling 3 dice?______. 4 dice?_____. N dice?______.

7. The Kampus Komputer Dating Service has cards for 250 men and 430 women. How many different dates can it arrange?

8. Consider the following results: 1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120. Can you determine the rule that was used to get these results?

9. Given a set of integral measures from 1 to 10, a student randomly draws three measures. What is the probability that these measurements could be used to form the sides of a triangle?

10. Draw some rectangular arrays made of squares and some parallelopiped arrays made of cubes, or use some square and cubic blocks to make rectangular or parallelopiped shapes respectively. Then give the number of squares or cubes in the array and show how you determined your answer to each case. (Make sure you try some square and cubic arrays.)

11. Select a set of five cubes of different color and place numerals on the faces as specified below:

    a. If two cubes are used, how many two digit numerals can be represented if you, choosing from the numerals 1 to 9, choose two digits and write each digit on three faces of each cube?

       Choose three digits and write each digit on two faces of each cube.

       Choose six digits and write each digit on one face of each cube.

What generalization can you make about determining the number of 2 digit numerals when choosing 2 cubes.

b. If three cubes are used, how many three digit numerals can be represented if you, again choosing from the numerals 1 to 9, choose two digits and write each digit on three faces of each cube?

Choose three digits and write each digit on two faces of each cube.

Choose six digits and write each digit on one face.

What generalization can you make about determining the number of numerals with n number of cubes?

II. PERMUTATIONS

Exercises

1. If you "box" (take all arrangements of the digits in the number) a number the agent must pay you no matter how the number falls. Only three digit numbers from 111 to 999 can be played.

   (a) If you box 591, what numbers will win?

   (b) Write down the numbers that cannot be boxed. In other words, what numbers will be the same when boxed?

   (c) Write down the numbers that can be boxed in three ways.

   (d) What is the probability of "hitting" if the following numbers are boxed:

   a) 591 b) 323 c) 555

   (e) Why are all the probabilities different in problem d?

2. Two chromosome Human Problems:

Father	Mother
SH	SH

The father and mother of a prospective two-chromosome child each have one sickle cell chromosome and one healthy chromosome. What is the probability that the child will be born healthy without a sickle cell chromosome?

Two chromosome Human Problem:

Father	Mother
SH	SH

The father of a prospective child has one healthy chromosome and one sickle cell chromosome, the mother has two healthy chromosomes. What is the probability that the child will be born with two healthy chromosomes?

3. In the following exercises, the duplicate letters are considered indistinguishable.

   a) How many distinct arrangements of the letters MOO can you find using all three letters?

   b) Same question for the letters in the word MOON.

   c) Same question for NOON.

   d) MOONO?

   e) NOONO?

   f) NOOOO?

   g) Can you generalize?

4. Given the letters of the work EXCEED, how many distinct arrangements can be made using all six letters?

   How many of the above arrangements from the word EXCEED will begin with an E? How many will end with an E? How many of the words will begin and end with an E? How many of these arrangements will begin with an E or end with an E?

   Can you associate the questions and answers with Venn diagrams and the related probability of the sets.

5. In how many ways can 2 (then 3, 4, 5, etc., r) people choose seats in a room containing 2 chairs? 3 chiars? 4 chairs? 26 chairs? N chairs?

6. The Greek alphabet has 24 letters. How many fraternity names 3 letters long can be formed?

7. How many different arrangements can be made from the letters of the word Mississippi using all the letters each time?

8. If 3 (then 4, 5, etc., N) cubes of different colors are placed in a line, how many different arrangements by color are possible?

9. A man has a combination lock with 40 numbers on it. He forgets the combination but remembers that it takes 4 different numbers to open it. Would he be wise to try all the combinations in order to find the one that opens the lock?

10. After a long meeting, a group of thirty students decided to have a pizza party. One of the students took on the job of ordering the pizzas for the crowd. The local pizza shop had five ingredients from which to make pizzas. They were mushrooms, onions, pepperoni, sausage, and anchovies. Of course he and the basic pizza from which he made any order which was handed to him. Each of the pizza sold for $2.00. The student who had taken on the job of ordering the pizzas became very confused with all the various orders from the crowd and finally said in a loud voice, "I am going to order one of every kind of pizza that they make, and you can just take your choice." The question is -- Will the young man have enough money to pay for the pizzas?

III. COMBINATIONS

Exercises

1. Suppose you are walking home from school and you must go 2 blocks east and 2 blocks north. If you walk sometime north and sometime east, changing directions (at intersections only) whenever you feel like it, how many different ways do you have to walk home? _______ (It might be useful to make a diagram and map out the different ways possible.)

   Suppose you are walking to a friend's house who lives 5 blocks east and 3 blocks north. If you walk to his house in the same manner you walked home from school, how many different ways do you have to walk to your friend's house? _______

2. Take three cubes of different colors (say a red cube, a blue cube, and a green cube) and place them in a box. No matter how they are arranged in the box, they are contained in the box. How many possibilities are there for the colors of the cubes contained in the box? _______

   Suppose one cube is taken out, what are the possibilities for the colors of the cubes remaining in the box? _______

   Suppose 2 cubes are taken out, what then are the possibilities? _______

   Suppose all three cubes are taken out, what are the possibilities? _______

3. Let's change the situation slightly. Instead of three cubes we have 8 cubes of different colors (the colors might be red, green, blue, purple, gold, black, orange and pink). In how many possible ways by color can three cubes be contained in the box? _______

   In how many ways can 4 cubes be contained in the box? _______
   5 cubes? _____ 6 cubes? _____ 7 cubes? _____ 8 cubes? _______

•4. A local business man has offered to send three seniors and two juniors to Europe on an educational tour. From a group of seven seniors and twelve juniors, how many different ways could this selection be made?

IV. PROBABILITY

Exercises

1. Last night the weatherman said that the probability of rain today is 20%. What did he mean?

    a) Did you bring your raincoat today? Why or why not?
    b) If he had said that the probability of rain was 70% would you have planned an outdoor picnic? Why or why not?

2. Create situations to represent the following:

    a) an event with several outcomes such that the probability of each outcome is the same.

    b) an event with several outcomes such that the probability of the outcomes are not all the same.

3. What do you think the probability of each of the following events is:

    a) an event that is sure to happen.
    b) an event that cannot happen.

4. Place the following known number of chips in boxes numbered 1 to 4, and shake each box well.

    Box 1 - 10 yellow and 1 red chip
    Box 2 - 8 green and 2 red chips
    Box 3 - 15 white and 2 red chips
    Box 4 - 15 orange and 3 red chips

    a) What are your chances (is the probability) of your drawing a red chip from box 1? Good? Fair? Poor?

    b) Are your chances better in one box than another? If so, in which box do you have the best chance of drawing a red chip?

    c) Arrange the boxes from left to right according to your chances of winning.

    d) Test your predictions by actually drawing from the boxes.

e) Assume that the boxes are not numbered. One box is chosen at random, and one chip is drawn from the box. What is the probability that the chip drawn is a red chip?

f) If two boxes are chosen at random and a chip is drawn from each box, what is the probability that a red chip and a white chip are drawn?

5. Suppose you flip a coin 10 times. How many of these times would you expect to get heads? _____

Now actually perform the experiment, that is to say, flip the coin 10 times and record the resulting number of heads. _____ Did it come out just the way you expected? Probably not ... and that doesn't necessarily mean that your guess is wrong. Remember, probability can only talk about what happens "in the long run" ... "on the average".

Try the experiment twice more, that is, flip the coin 10 more times, and then 10 again, and record the results (total number of heads each time) ______ ______ Were either of these any closer to your prediction, or farther away?

Suppose you added together the totals of your 3 experiments to find the fraction of the total number of flips that are heads. Would you expect this fraction to be closer to your prediction? Do this and record your result. _______ Is it closer to the prediction than the individual fractions?

Think about what you're doing as you figure your grand total and the corresponding fraction of heads ... Heaven forbid that anybody should just add the individual fractions together or something like that.

Is your fraction of heads (call this the experimental probability) for the 30 flips was closer to your prediction (call this the theoretical) than most of the ones for 10 flips.

When you bring this to class we will take everyone's totals, and add them all together and figure out the fraction of heads (experimental probability) for the class. Would you expect this to be closer to the theoretical probability than most of the fractions given by each person?

6. One die: What is the empirical (then theoretical) probability of getting a four in one toss?

7. Two dice: What is the empirical (then theoretical) probability of getting a total of five in one toss?

8. A bag with red and green poker chips: What is the empirical (then theoretical) probability of getting a green chip on the first draw?

9. A bag with blue, yellow, green, orange and red poker chips: What is the empirical (then theoretical) probability of getting a blue chip on the first draw?

V. INDEPENDENT AND DEPENDENT EVENTS

Exercises

1. Teams 1 and 2 are in contests. The probabilities that each will win are .6 and .4, respectively. Find the probability that both will win if:

    a) they are in separate contests.
    b) they are in the same contest.

2. In a track meet Sam, Henry and Mark are contestants in three different events. The probabilities that each will win first place are 5/8, 1/2 and 3/5 respectively.

    a) Find the probability that all three will win first place.

    b) Find the probability that Sam or Mark will win first place.

    c) Find the probability of each being the only one to win a first place.

3. Three blue books, five red books and two white books are on a shelf. A man selects two books at random to take on a vacation and draws them one at a time from the shelf.

    a) Find the probability that the first book drawn will be blue and the second white.

    b) Find the probability that the books will be drawn in either order red, white or blue, red.

4. Experiment: Roll a die until either a 4 or a 6 appears.

    Question: What is the probability that the 4 appears before the 6?

    Look at the question and think about it ... without doing any severe calculations, take a guess as to what the probability should be. Hint: think about which one would be more likely on a fair die. We will now check this guess ... if the guess seems very logical to you, checking it may seem unnecessary, but even in this case, the methods will be useful later.

    Could this experiment end after just 1 throw?
    Could this experiment last for 2 throws?
    Could it last for more than 5 throws? How?
    More than 20 throws? How?

By now you have perhaps convinced yourself that the-sample space in this experiment is infinite ... there is no limit to the number of throws that this experiment might require if neither the 4 or the 6 comes up for a long time. Since this seems too complicated to compute directly, let us try to add up the probabilities of a series of disjoint events which will add up to what we're interested in.

It is sometimes difficult to know how to get started. Let's take a sample outcome to the experiment which, we hope,will help us see what's important and what's not. Suppose the numbers in a certain run of the experiment show up like this: 1 5 3 1 4. Is this outcome in our event? (Which showed first, a 4 or a 6?) How about this one: 3 2 5 1 4? What is the probability of the second outcome?

Notice that they are very similar ... in both cases, a 4 showed up on the fifth throw. We would go crazy if we tried to figure out separately each of the possible outcomes, even for the case where it goes 5 throws. Actually though, they're all pretty much the same, so we can lump them together.

This is our plan, then. We will lump the outcomes together according to the number of throws required; then add up the resulting probabilities to get the answer.

We need some information:

P (1, 2, 3, or 5 on a given throw) = ________
P (4 on a given throw) = ________

Fill in the following chart:

number of throws in the outcome (n)	P(1,2,3, or 5 on every throw but the last)	P(4 on the nth throw)	P(experiment lasts n throws and ends in a 4

Notice that becasue of the independence, the last column is gotten by multiplying together the results of the first two columns.

Also, it is these numbers in the last column that we have to add up. The column is not complete, of course, we just put down enough to help see a pattern. Do you see one?

Now, use this pattern to add up the series and get the results:
P(a 4 comes before a 6) = ________

5. Experiment:
Ground Rules - Roll a pair of dice. All numbers refer to the sum of the faces showing on the dice.

If you roll a 7 or an 11 you win.
If you roll a 2, 3, or a 12 you lose.
If you roll any other number, N, you must keep rolling the dice until one of the following things happen:

1. you get N again ... in which case you win or
2. you get a 7 ... in which case you lose.

This is a fairly complex game ... too much so to be able to expect to figure out the probability of winning all in one step. Hence, we break up the problem into smaller ones by trying to figure out the probability of each of the various ways of winning ... if we do this right then there will be no overlap ... that is, they will be disjoint events, and, as you know, we may then just add up the various probabilities to get the answer we want. The question is: How to do this.

Notice that the form the game takes is pretty much determined by what happens on the first move, so this suggests a good possibility of how to divide the problem up: namely, according to what you get on the first throw.

Remember that we want the probability of winning the game, so there are certain first throws we can ignore because we can't possibly win if we get them. What are they?

It will help us to organize our thinking if we make a chart of the various probabilities of winning with the various first moves ... even though we won't be able to fill it in completely for a while.

Event	Probability
P (first roll is a 4 and we win)	
P (first roll is a 5 and we win)	
P (first roll is a 6 and we win)	
P (first roll is a 7 and we win)	
P (first roll is an 8 and we win)	
P (first roll is a 9 and we win)	
P (first roll is a 10 and we win)	
P (first roll is an 11 and we win)	

Now look at the chart to check that this is a reasonable system: First notice that the union of these events is the event "we win" ... there are no other ways to win. Why is this?

Now check: Are these events pairwise disjoint ... that is to say, is there any overlapping between any two of them?

Look over the chart again ... you should be able to find a couple of entries that you can calculate with a little bit of thought right now. Do this and fill in those entries. (You are able to do this since certain first rolls automatically end the game and you only have to worry about what happened on that one roll.)

Now lets look at the rest of the entries on the chart ... they all fall into pretty much the same pattern, so let's look at the first one as a sample so we can see what's happening. If you have trouble seeing what this is going to involve, then it might be a good idea to get a couple of dice and try playing the game a few times to see how it goes.

Look again at that first entry ... it is a compound probability... of the general form P(A and B) where A and B are two events. This is very promising because if A and B are independent events, Then we have an easy way to calculate P(A and B). What is it in terms of P(A) and P(B), P(A and B) = ? Therefore, the $64 question right now is this:

    Are the events A = "The first roll is a 4"
and   B = "We win"   independent???

At this point, you might want to consult with the teacher on the answer and results that you have gotten so far ....

Before we try to repair the fault, and fortunately it won't be hard, let's see why they were not independent. For them to be independent would mean that knowing that you rolled a 4 the first time would have no effect on your chances of winning ... in other words, you would have just exactly as good a chance of winning with a 4 as you would at the beginning, before you knew what you were getting on that first roll. Well, ... at the very least that is not obvious ... actually it turns out not to be true ... a 4 is a rather hard point to make ...

Look at the event B = "We win". This event certainly depends on the outcome of the first throw ... but we know what that is in this case ... a 4. We can be more explicit: Exactly how do you win when your first thorw is a 4? Make up a statement to describe how to win.

Call the statement you made up the even C, and notice that if your statement is correct, then A and B is the same event as A and C, because C is just a more careful description of "we win".

Furthermore, your statement C should only refer to the throws after the first one ... to insure that the events A and C are independent.

So: P(A and B) = P(A and C) = P(A) XP(C)

What is P(A)?

P(C) is definitely a harder problem ... try to work this one as you worked the "4 before 6" problem before.

Namely: Divide the event C into a series of categories:

P(4 on second throw)
P(4 on third ... and ... neither 4 nor 7 on second)
P(4 on fourth and neither 4 nor 7 on second or third)
P(4 on fifth and neither 4 nor 7 on second through fourth)
P(4 on sixth and neither 4 nor 7 on second through fifth)
and so on ...

Now, calculate each of these ... remember ... each of the throws is independent of the others, so we may calculate by multiplying the appropriate probability for each throw.

Now ... what are these? P(4 on a given throw) = ?
P(neither 4 nor 7 on a given throw)= ?
Use these to calculate each of the above probabilities.

Notice that these are disjoint ... there is no way that two of the categories could happen at once.

Again, we only wrote down enough of the categories to get a pattern ... when we add all of these together, it will actually be an infinite sum.

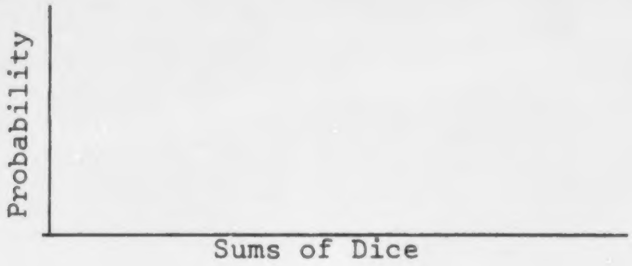
After you have completed this, go through the same process with each of the other probabilities in the chart. Remember to first modify the even "we win" as before.

When you have the whole chart completed, the final answer is just the sum (they were disjoint) of the various probabilities in the chart.

6. Suppose a young married couple has been told that the probability that any one of their children would have some "wonder" trait was 1/8. In the event that they have five children, what is the probability that: a) all will have the trait, b) four will have the trait, etc. What should the sum of all possible be?

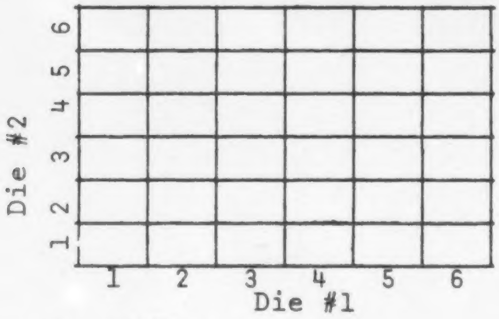
VI. SAMPLE SPACES

1. Analyze the empirical results of a pair of standard dice. First roll the dice 20 times and record the results in graph form.



Now analyze the expected (theoretical) results for a pair of standard dice. Can you predict what the theoretical probability curve will look like before doing the problem?

A Chart Of All Possible Sums



a) What are the different sums?

b) What is the probability of each sum?

Now construct a graph of the theoretical probability of getting any given sum.

2. Analyze a pair of dice with the numbers 4, 5, 6, 7, 8, 9 on their faces. What will the theoretical probability graph look like?

3. Analyze the system of a penny and a nickel. Can you draw a probability graph for this system?

VII. WHAT'S ON THE DICE?

The following numbers were the sums of the numbers on the upward faces of the two dice after each roll.

7	18	7	26	69	12	12	57	16	69	26	11	7	11	54	69
22	8	11	18	10	31	11	22	16	33	27	37	18	12	33	16
54	33	18	22	35	39	31	19	7	7	21	20	16	82	69	22
39	54	44	69	22	7	26	27	65	8	16	41	20	10	21	22
65	20	8	14	16	54	7	10	7	18	37	16	18	10	29	20
57	69	8	29	57	35	29	39	31	39	54	65	5	7	5	37
37	7	59	33	44	12	82	35	5	33	57	33	59	5	19	22
16	5	41	31	8	31	33	16	26	20	33	33	33	18	20	22
44	31	14	35	16	16	26	12	41	39	8	33	41	20	29	82
22	54	39	12	26	10	54	10	69	35	82	44	54	35	16	57
22	10	10	33	82	33	39	10	20	22	16	18	54	5	5	10
26	39	16	10	29	12	37	5	10	57	57	44	22	14	26	29
39	5	54	29	65	16	31	19	27	54	82	26	21	82	18	82
26	33	21	57	5	44	37	16	21	11	41	31	21	16	8	21
82	20	41	16	19	44	16	31	7	41	26	26	31	10	5	65
31	8	54	16	8	5	7	65	10	65	44	59	26	33	41	18
10	10	69	59	22	7	16	54	59	37	27	31	19	69	10	57
10	26	82	82	14	12	54	21	65	19	59	41	37	82	59	69
29	14	5	8	37	57	22	5	11	26	10	12	41	19	35	16
12	16	37	21	33	65	44	12	57	10	16	26	21	69	31	39
16	12	29	54	41	82	8	54	21	18	19	37	41	44	29	22
16	54	82	54	11	5	16	26	26	20	16	54	16	22	20	10
54	22	7	20	31	19	21	5	33	29	35	27	26	16	7	82
57	35	39	12	29	21	26	19	82	54	8	7	20	26	11	20
18	16	31	18	44	10	44	7	22	7	54	31	44	5	8	18

Question: Which numbers were on the faces of each of the dice?

Exercises

1. If you had ten coins with the numbers 1 and 2 on the faces, what is the probability of getting a sum of 15? 13? 20? 24? etc.

2. If the president is predicted to win 60% to 40%, what is the probability that three of the first four people picked at random will vote for him?

3. Suppose a rifleman has a probability of 0.05 of hitting the target. What is the probability that in 10 shots he never hits the target?

4. Suppose that the normal rate of infection of a certain disease is 25%. What is the probability that in a family of 6 members all will catch this disease?

5. Probability of finding an albino child is 0.1. What is the probability that in a family of 4 children at least 2 will be albino?

6. A baseball player's batting average is 0.7. What is the probability that he gets at least 3 hits in 5 times at bat?

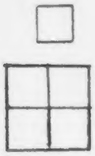
7. Suppose that, in flight, airplane engines fail with probability 0.001 independently from engine to engine and that a plane makes a successful flight if at least half of its engines run. What is the probability that a 4-engine plane will fail to make a successful flight?

- TRIGONOMETRY -

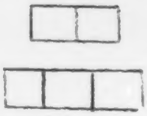
I. SIMILARITY

1. Using the square pieces from the set of basic shapes make the following pairs of figures:

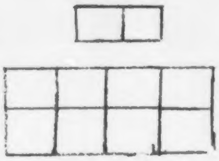
(1)



(2)



(3)



   a) By comparing the figures in each pair, decide whether they have the same shape or not.

   b) Are the figures in each pair similar to each other? Explain.

   c) Make a third figure that will be similar to the second figure in each pair.

   d) Are all squares similar? What about all rectangles? Why?

   e) What attributes must be investigated in determining similarity among figures?

2. Using the equilateral triangular pieces, make three different size triangular figures.

   a) Which attributes remained the same for all three figures? Which attributes changed?

   b) Is it possible to make a triangular figure that is not of the same shape as one of the given pieces?

   c) Are all three triangles similar to each other?

   d) Can you conclude that all equilateral triangles are similar? Why or why not?

3. Now, using the set of right-triangular pieces, make three different size triangular figures.

    a) What attributes remained the same for all three figures?

    b) Are all three triangules similar to each other?

    c) Do all right triangles have the same shape?

    d) Are all right triangles similar to each other? Talk about this.

    e) What determines similarity among triangles?

4. How does the idea of similarity partition triangles into equivalence classes?

Exercises

1. The legs of a right triangle are 3 and 4 inches respectively. In a similar right triangle the shortest side is 5 inches. What are the lengths of the other sides?

2. The vertices of three right angled triangles are given by the following sets of Cartesian coordinates:

    $A = \{(-6, -6), (6, 10), (6, -6)\}$
    $B = \{(-2, 6), (2, 10), (4, 8)\}$
    $C = \{(-4, 6), (2, -2), (-4, -2)\}$

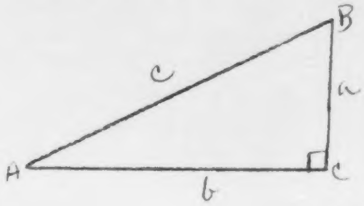
    Which two triangles are similar? Verify your answer.

3. When a man 6 feet tall casts a shadow 4 feet long, a tree casts a shadow 15 feet long. What is the height of the tree?

4. The lengths of the sides of a triangle are 7, 10 and 12 inches. If the length of the longest side is increased by 3 inches, how much should the other sides be increased so that the new triangle is similar to the old?

II. SPECIAL TRIANGLES -

Using a ruler and protractor, do the following experiments and record the data in the given table. Do drawings on notebook paper. When labeling your triangles use this one as a guide.



1. Draw the following 45° right triangles:

   a) a = b = 2 inches
   b) a = b = 3 inches
   c) a = b = 4 inches
   d) a = b = 4 centimeters

2. Draw the following 30° -60° right triangles:

   a) a = leg opposite 30° angle is 3 inches
   b) b = leg opposite 30° angle is 5 inches
   c) a = leg opposite 30° angle is 5 centimeters
   d) b = leg opposite 60° angle is 6 centimeters

	Length of Side a	Length of Side b	Length of Side c (hypotenuse)	R_1 Ratio of a/c	R_2 Ratio of b/c	R_3 Ratio of a/b	R_4 Ratio of b/a	R_5 Ratio of c/b	R_6 Ratio of c/a	Number of Degrees in ∠ A	Number of Degrees in ∠ B	Number of Degrees in ∠ C
1a												
b												
c												
d												
2a												
b												
c												
d												

3. List as many observations as you can from the table.

4. Suppose we name the ratios as follows:

$\text{sine} \angle A = \frac{a}{c}$ $\quad$ $\text{cosecant} \angle A = \frac{c}{a}$

$\text{cosine} \angle A = \frac{b}{c}$ $\quad$ $\text{secant} \angle A = \frac{c}{b}$

$\text{tangent} \angle A = \frac{a}{b}$ $\quad$ $\text{cotangent} \angle A = \frac{b}{a}$

   a) What would be the ratios for <B?

   b) Complete the following table.

Angle	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
30°	1/2					
60°						
45°						

Let us consider only those triangles that are right triangles, i.e., triangles on which $(m(a))^2 = (m(c))^2$ where a, b, c are the sides of the triangles (Note: m(a) represents the measure of side a, etc.)

Make a list of the measures of the sides of some such triangles* (one example has been given).

m(a)	m(b)	m(c)
3	4	5
---	---	---
---	---	---
---	---	---
---	---	---
---	---	---

---

*Students who have access to a computer might try writing a computer program for this problem.

1. Is the ratio $\frac{3}{11}$:33 equivalent to 1:11?

   Is the ratio $\frac{3}{2}$:6 equivalent to 3:12?

   What is the simplest form of these ratios?

2. Suppose N represents any number except zero. Will all the ratios expressed by N:4N be equivalent? If so, what will be the simplest form for the set of equivalent ratios? Experiment with a few replacements for N. Use fractions and decimals as possible replacements for N.

II. Proportion

1. A mathematical sentence that says two ratios are equivalent is called a proportion. Explain why each of the following is, or is not, proportion.

   a) 7 - 5 = 2
   b) 8:3 = 16:6
   c) 16 ÷ 4 = 2 + 2
   d) 4 is to 3 as 8 is to 6

2. Solve these proportions:

   a) 1:2 = n:14
   b) x:42 = 4:7
   c) x:4.2 = 3:1.4

III. RELATIONS AMONG TRIGONOMETRIC FUNCTIONS

Exercises

A. Recall and observe further the entries in your table for the 45° and 30 -60° right triangles.

Angle	Sin	Cos	Tan	Cot	Sec	Csc
30°						
60°						
45°						

1. Compare the actual values of these ratios.

2. Which of the following statements are true in general from your observations:

    a) $\mathrm{Sin}\ A < \cos A$

    b) $\mathrm{Sin}\ A > \cos A$

    c) $\mathrm{Sin}\ A = \cos A$

3. Make similar comparisons for the other ratios.

4. If you were asked to list the ratios of a particular angle in order of size from the smallest to the largest, how would such a listing look?

5. Using values from the table, perform some operations on the ratios. Here are some possibilities:

    a) $\sin A \times \frac{1}{\cos A}$

    b) $\cos A \times \frac{1}{\sin A}$

    c) $1 \div \sin A$ or $1 \div \cos A$ or $1 \div \tan A$

    d) What conjectures would you make concerning these products and quotients?

    e) Put your findings in general form.

6. Try adding or subtracting two ratios. Add or subtract squares of two ratios.

    Can you use your results to make further conjectures concerning relations among trigonometric functions?

B. Trigonometric Identities

Exercises

1. Trigonometric identities

1. $\text{Sin}A = \frac{1}{\text{csc}A}$ 2. $\text{Cos}(A) = \frac{1}{\text{sec}A}$ 3. $\text{Tan}A = \frac{1}{\text{Cot}A}$

4. $\text{Cot}A = \frac{1}{\text{Tan}A}$ 5. $\text{Sec}A = \frac{1}{\text{Cos}A}$ 6. $\text{Csc}A = \frac{1}{\text{Sin}A}$

7. $\text{Tan}A = \frac{\text{Sin}A}{\text{Cos}A}$ 8. $\text{Cot}(A)\ \frac{\text{Cos}A}{\text{Sin}A}$

9. $\text{Sin}^2A + \text{Cos}^2A = 1$ 10. $1 + \text{Tan}^2A = \text{Sec}^2A$

11. $1 + \text{Cot}^2A = \text{Csc}^2A$

We use these eleven identities in establishing the truth for other trigonometric identities.

1. prove: $(1-\sin^2A)\ (1 + \tan^2A) = 1$

2. prove: $\sin(A) \times \text{Cot}(A) \times \sec(A) = 1$

2. Verify the following identities:

1. $\sin^2A - \sin A.\text{Cos}A + \text{Cos}^2A = 1 - \sin(A).\text{Cos}(A)$
2. $\text{Cos}^2A \times (1 + \tan^2A) = 1$
3. $\text{Cos}^2A - \text{Cos}^2A = \text{Cos}^2A \times \tan^2A$
4. $\tan^2A \times \text{Cos}^2A + \text{Cot}^2A \times \sin^2A = 1$
5. $\sec^2A.\text{Csc}^2A = \sec^2A + \text{Csc}^2A$

3. Simplify the following trigonometric expressions:

1. $\tan^2A \times \text{Csc}^2A \times \text{Cot}^2A$ 2. $\tan A + \text{Cot}\ A$

3. $\sin A/\text{Csc}A + \text{Cos}A/\sec A$ 4. $\sin A \times \text{Cos}A.\tan A + \sec A$

5. $(1-\text{Cos}A) \times (1+\sec A \times \text{Cot}A)$

C. Sums and Differences of Angles - Exercises

Exercises

1. Using the angle sizes 0°, 15°, 30° and 45°, tell whether or not the formulas below are true:

    a) $\text{Sin }(A+B) = \text{Sin } A + \text{Sin } B$
    b) $\text{Sin }(A+B) = \text{Sin } A.\text{Sin } B + \text{Cos}A.\text{Cos}B$
    c) $\text{Sin }(A+B) = \text{Sin}(A) \text{ X } \text{Cos}(B) + \text{Sin}(B) \text{ X } \text{Cos}(A)$
    d) $\text{Sin }(A+B) = \text{Sin}(A) \text{ X } \text{Sin}(B)$

2. Using the natural triginometric ratios determine which of the following are true:

    a) $\text{Sin }(10^\circ + 10^\circ) = \text{Sin }(10^\circ) + \text{Sin }(10^\circ)$
    b) $\text{Sin } 30^\circ = \text{Sin } 10^\circ \text{ X } \text{Cos}(20^\circ) + \text{Sin }(20^\circ) \text{ X } \text{Cos}(11^\circ)$
    c) $\text{Sin } 23^\circ = \text{Sin}(11^\circ) \text{ X } \text{Cos}(12^\circ) + \text{Sin}(12^\circ) \text{ X } \text{Cos }(11^\circ)$
    d) $\text{Sin } 45^\circ = \text{Sin}(15^\circ) \text{ X } \text{Cos }(30^\circ) + \text{Sin } 30^\circ \text{ X } \text{Cos }(15^\circ)$

3. Use your trigonometric ratios to tell whether or not the formula is true for the values given.

    a) $\text{Sin }(A-B) = \text{Sin}(A) - \text{Sin}(B)$
    b) $\text{Sin }(A-B) = \text{Sin}(A) \text{ X } \text{Sin}(B) - \text{Cos}(A) \text{ X } \text{Cos}(B)$
    c) $\text{Sin }(A-B) = \text{Cos}(A) \text{ X } \text{Cos}(B) - \text{Sin}(A) \text{ X } \text{Sin}(B)$
    d) $\text{Sin }(A-B) = \text{Sin}(A) \text{ X } \text{Cos}(B) - \text{Sin}(B) \text{ X } \text{Cos}(A)$

4. Using the natural trigonometric ratios, determine which of the following are true.

    a) $\text{Sin}(45-30^\circ) = \text{Cos}(30^\circ) \text{ X } \text{Cos }(45^\circ) + \text{Sin}(30^\circ) \text{ X } \text{Sin}(45^\circ)$
    b) $\text{Sin}(45^\circ-30^\circ) = \text{Sin}(45^\circ) \text{ X } \text{Cos}(30^\circ) - \text{Cos}(45^\circ) \text{ X } \text{Sin}(30^\circ)$
    c) $\text{Sin}(15^\circ-15^\circ) = \text{Sin}(15^\circ) \text{ X } \text{Cos}(15^\circ) - \text{Cos}(15^\circ) \text{ X } \text{Sin}(15^\circ)$
    d) $\text{Sin}(45^\circ-15^\circ) = \text{Sin } 45^\circ \text{ X } \text{Cos } 15^\circ - \text{Cos } 45^\circ \text{ x } \text{Sin }(15^\circ)$

5. Using the angle size 0°, 15°, 30° and 45°, determine if the following are true.

    a) $\text{Sin}(A-B) = \text{Sin}A.\text{Cos}B - \text{Sin}B\ \text{Cos}A$
    b) $\text{Sin }(A+B) = \text{Sin}A.\text{Cos}B - \text{Sin}B\ \text{Cos}A$
    c) $\text{Cos }(A-B) = \text{Sin}^2A - \text{Cos}^2A$
    d) $\text{Cos }(A+B) = \text{Cos}^2A - \text{Cos}^2B$

D. Half-angle formulas

We have determined that:

$\text{Cos}(A+A) = \text{Cos}^2A - \text{Sin}^2A = 1 - 2\,\text{Cos}^2A.$

Since this formula does hold for every angle A, it must necessarily hold for half of that angle. In other words, replacing A by A/2 in the above formula, we get

$\text{Cos}(A) = 1-2\,(\text{Sin}(A/2))^2$ This formula is equivalent to

$\text{Sin}(A/2) = \sqrt{\frac{1-\text{Cos}A}{2}}$ Similarly, we may determine $\text{Cos}(A/2) = \sqrt{\frac{1+\text{Cos}(A)}{2}}$

a) Without using a table, determine the values of the trigonometric ratios for the classes of right triangles identified by the following acute angles:

(1) 7 1/2° (2) 3 3/4° (3) 11 1/4° (4) 33 3/4°

Use a table of trigonometric ratios to determine the six ratios for each of the following angles:

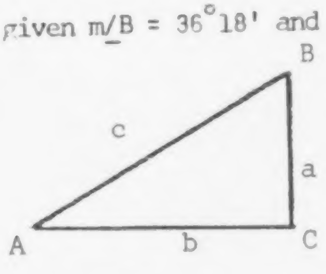
(a) 19° (b) 27° (c) 33 1/2° (d) 42°17'

E. Solving Triangles

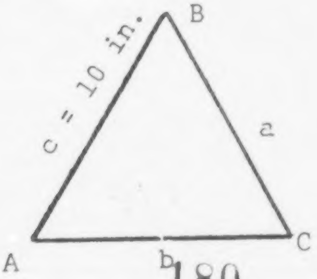
1. Triangulation:

(a) Solve the right triangle ABC, given m/A = 36°18' and m(a) = 46.94 inches.

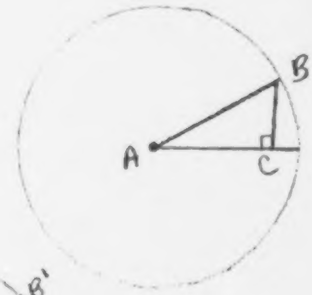
(b) Solve the right triangle ABC, given m/B = 36°18' and m(b) = 46.94 inches.



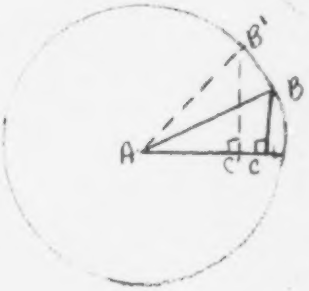
(c) Solve the triangle ABC, given m/A = 60° = m/B, and m(c) = 10 inches.



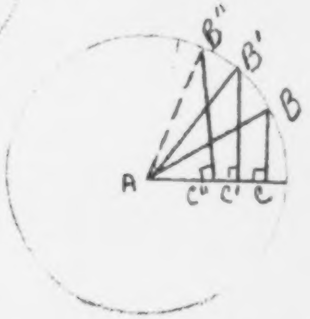
IV. RANGE OF TRIGONOMETRIC FUNCTIONS
(Sin and Cos)



In the circle to the left, the radius is 1 unit. AB is a radius and therefore is 1 unit long. Angle A is some acute angle.



Suppose we increase the size of angle A (do not make m<A greater than 90°, however) so that we can draw another triangle A'B'C'.



Increase <A again (do not make m<A greater than 90°) so that we can draw another triangle A B" C". You should notice that $\overline{AB}$, $\overline{AB'}$ and $\overline{AB''}$ are all radii of the circle.

1. a) How can the sin of angle A in each of the triangles be represented?

   b) What happens to the sin of angle A as the size of the angle increases?

   c) Explain how you are able to tell by using the diagram that this will always be true as an angle increases from 0 to 90 degrees.

   d) What do you think sin 0° is? Why? Sin 90°? Why?

2. a) How can the cos of angle A in each of the triangles be represented?

   b) What happens to the cos of angle A as the size of the angle increases?

   c) Use the diagram to explain why this will always be true about the cos of an angle as it increases from 0° to 90°.

   d) What do you think cos 0° is? Why? cos 90°? Why?

V. GRAPHS OF TRIGONOMETRIC FUNCTIONS

Exercises

Even though we are not carefully defining units of angular measure, we briefly indicate the approximate size of one unit of angular measure - the radian. The real number symbolized by π is approximately 3.14. The radian is used to measure angles, and 1 radian is approximately 57.3°; π radians = 180°. Here, we are not concerned about how radian measure arises. When we write sin x without a degree symbol we shall mean "sin(x radians)", and cos x shall mean "cos (x radians)". If degree measure is intended, let's agree to consistently use the degree symbol; for example, sin 30°, rather than sin 30.

Exercise 1 Obtain the degree equivalent of

a) $\frac{3\pi}{2}$

b) $\frac{\pi}{5}$

c) 7π

d) $\frac{\pi}{4}$

e) $\frac{4\pi}{3}$

Exercise 2 Obtain the radian equivalent of

a) 70°

b) 360°

c) 30°

d) 45°

e) 900°

f) 460°

Following are three groups of equations, labeled A, B, and C. For each equation in each group set up a table of values then graph the equation (on graph paper). Equations from the same group must be graphed on the same graph sheet using the same coordinate frame. Be sure you plot points sufficiently close to insure a good approximation to the graph of each equation.

Here, we will assume that x is a real number between -4π and 4π. Actually, we are only interested in the portion of certain trigonometric curves over the part of the real axis bounded below by -4π and above by 4π.

A.

a) $y = \sin x$

b) $y = 2 \sin x$

c) $y = 3 \sin x$

d) $y = 4 \sin x$

B.

a) $y = \sin x$

b) $y = \frac{1}{2} \sin x$

c) $y = \frac{1}{3} \sin x$

d) $y = \frac{1}{5} \sin x$

C.

a) $y = \sin x$

b) $y - -\sin x$

c) $y = -2 \sin x$

d) $y = -\frac{1}{2} \sin x$

e) $y = -3 \sin x$

You may use any observation about the behavior of the graphs to facilitate the sketching of the graphs - Do not hesitate to do this.

Now compare all graphs with the graphs of $y = \sin x$.

a) Can you find a general form for the equation in each group separately? Identify the changing parameter in each case.

b) Can you find a general form which holds for all groups? Identify the changing parameter.

c) What effect does the changing parameter have in each case?

d) Looking at the general equation in (b), would you indicate the symbol representing the changing parameter?

e) Comment on its over-all effects on the shape of the graph.

f) Can you think of the equations above as belonging to a class, category, or family in some sense? How or why?

Following are two equations, each of which belongs to a whole family of equations which are alike in a certain respect. Look at the previous groups studied and, first, be sure that you do see that they belong together in some sense. Now see if you can think of some more equations that belong to the same family as does each of the equations below. Then following the pattern of analysis above, see if you can determine what the graphs of each family should look like. The equations are:

1. $y = \sin x$
2. $y = \sin (x + \pi)$

Based on what you have done so far, what effect does each of the following parameters have on the shape of the graph of $y = n \sin(mx+k)$?

a) a b) m c) k

Can you quickly sketch graphs of the following?

a) $y = 2 \sin x$

b) $y = \frac{1}{2} \sin x$

c) $y = -3 \sin(x + \frac{\pi}{2})$

d) $y = 4 \sin(3x)$

e) $y = 3 \sin (\frac{1}{2} x)$

f) $y = 3 \sin(x - 3\pi)$

g) $y = -\frac{1}{3} \sin x$

Can you do a similar analysis for the graph of $y = a \operatorname{Cos} (nx + b)$?

Compare the graphs of the following pairs of equations:

1. $y = \sin x$ and $y = \operatorname{Cos} x$
2. $y = 2 \sin x$ and $y = 2 \operatorname{Cos} x$
3. $g = 3 \sin x$ and $y = 3 \operatorname{Cos} x$
4. $y = \sin(x + \frac{\pi}{2})$ and $y = \operatorname{Cos} (x + \frac{\pi}{2})$
5. $y = \operatorname{Sin} x$ and $y = \operatorname{Cos} (x + \frac{\pi}{2})$

What are your observations? Can you find a simple relationship between the sine and Cosine graphs? How would you express this mathematically?

If you are not sure, just imagine taking the cosine graph in each case and sliding it either up or down, left or right, or compressing it or expanding it, in any way you wish until it fits nicely over the corresponding sine graph for the equation on the left. Observe carefully what you need to do to bring about coincidence of the graphs.

Now use your knowledge of the sine function and the relationship between the sine and the Cosine functions to sketch, quickly, graphs of the following:

1. $y = 2 \text{ Cos } x$
2. $y = \text{Cos } 5x$
3. $y = \frac{1}{2} \text{ Cos } (x + \frac{\pi}{2})$
4. $y = - \text{Cos } 2x$

Functions like Cos x and sin x are called periodic functions. Is there something periodic about the days of the week? Days of the year? the earth's rotation about its axis? Can you think of some other phenomena which you think of as periodic in some sense? Now why do you suppose Cos x and sin x are called periodic? Look at each graph very carefully.

The graph of sin x is usually called the sine curve and the graph of cos x the Cosine curve.